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# Reduction and reconstruction of the dynamics of nonholonomic systems

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**Abstract.** The reduction and reconstruction of the dynamics of nonholonomic mechanical systems with symmetry are investigated. We have considered a more general framework of constrained Hamiltonian systems since they appear in the reduction procedure. A reduction scheme in terms of the nonholonomic momentum mapping is developed. The reduction of the nonholonomic brackets is also discussed. The theory is illustrated with several examples.

## 1. Introduction

The main goal of this paper is to carefully analyse the reduction and reconstruction of nonholonomic mechanical systems with symmetry. Our starting point is the general setting for constrained systems developed by Cantrijn *et al* [10]. As stated there, this framework provides a unifying model for the description of degenerate systems as well as of mechanical systems with nonholonomic constraints. This generality is not fictitious because in the reduction procedure of some particular nonholonomic systems we need to consider it.

The classical approach to nonholonomic mechanical systems is based on the method of Lagrange multipliers (see, for example, [38] for a comprehensive treatment). The geometric foundations of the theory were stated by Vershik and Faddeev [43, 44], and the subject has generated a great deal of interest since the fundamental work by Koiller [17]. At this moment, there are essentially three different, but related, approaches. A Hamiltonian approach, due to Bates and Śniatycki [2, 4, 40], which is based on the construction of an adequate bundle along which the constraints vanish and the equations of motion continue to be Hamiltonian; a Lagrange multipliers approach by Marsden *et al* [6, 18–20] which is a modern adaptation of the classical method; and a Lagrangian approach by de León and de Diego [22, 23] (see also [24, 25]) who worked on the tangent bundle and derived the equations of motion by explicitly constructing a vector field yielding the dynamics. A more general Poisson framework was considered by Marle [31,32]. The underlying affine differential geometry of nonholonomic systems has been investigated in [5, 14, 28, 29].

A mechanical system subject to constraints usually exhibits many symmetries, so in recent years there have been many attempts to adapt the well known symplectic reduction schemes for these systems. The main difficulty stems from the fact that, in contrast to the unconstrained case, the symmetry of a nonholonomic system does not generally produce a conserved quantity (moreover, if the constraints are nonlinear the energy is not, in general, a conserved quantity). Indeed, in [25] (see also [11, 40]) a Noether theorem was proved that gives a necessary and sufficient condition for a quantity to be conserved.

In [6], a nonholonomic momentum mapping which extends the standard one for unconstrained systems was proposed; in fact, we first have to identify the situation of the constraints with respect to the symmetries and then define, at each point, a subspace of the Lie algebra of the group of symmetries. The nonholonomic momentum mapping is then the restriction of the usual one, but pointwise. In [6], a momentum equation was given for the variation of the momentum along the trajectories of the system. In [7, 9, 10] the authors have derived a momentum equation in terms of the dynamics.

One of our aims in this paper is to perform reduction via the nonholonomic momentum mapping. With this purpose in mind, we introduce the notion of coadjoint representation and the isotropy group in this nonholonomic context. Our results are the natural extension of the symplectic reduction procedure, of course, with obvious differences and particular restrictions. In the kinematic case, these results cannot be adapted, due to the lack of a momentum map. In contrast, we have obtained new results in the case of horizontal symmetries by applying the theory developed for the general case.

In any case, a key point in this study is the fact that the equations of motion for a nonholonomic system are not Hamiltonian in the standard sense. This can be exhibited in several ways (see [23, 25]), but one clear piece of evidence highlighting this fact is that the evolution of the system cannot be described by using the standard Poisson bracket. Indeed, one has to define a new bracket on the constraint submanifold which gives the correct evolution of observables and, in particular, provides the equations of motion. This bracket does not enjoy the Jacobi identity, so it was called the nonholonomic bracket and, in a more general context, the almost-Poisson bracket (see [8, 15, 16, 26, 32]). The nonholonomic bracket was first considered by Eden [12, 13] and later rediscovered by van der Schaft and Maschke [42]. The relation between nonholonomic brackets and momentum mappings was exhibited in [7]. Nonholonomic brackets have been used in recent papers to obtain reduction procedures (see [2, 9, 18–20]), and they are widely used throughout this paper.

The reconstruction of the dynamics process for nonholonomic systems has been treated in [6]. Here, we present an exposition of some new results and generalizations on the subject in the context of the general framework for constrained systems. We also point out the similarities between both situations.

This paper can be summarized as follows. In section 2 we give a brief description of the general framework for constrained Hamiltonian systems developed in [10], with special emphasis on nonholonomic mechanics. The classification stated in [10], inspired by the paper of Bloch *et al* [6], is also reviewed in section 3. In section 4 we discuss the reduction scheme of the general case by means of the nonholonomic mapping mentioned above. We have derived three different points of view to tackle the problem and illustrate them rephrasing the example of the nonholonomic free particle in section 4.1.4. The kinematic case is considered in section 5, where we pay particular attention to the case of Čaplygin systems. After recalling the reduction procedure in section 5.1, we investigate the reconstruction process and obtain some nice results in section 5.2. The case of horizontal symmetries is the subject of section 6. We briefly review the reduction scheme and then develop a reconstruction process, paying special attention to nonholonomic systems. In section 7, we investigate the particular case when the bundle of 'admissible values' for the momenta is trivial. The main motivation for this treatment is to establish a well-posed reduction process in two steps, 'breaking' the symmetries to obtain first a horizontal case and, secondly, a purely kinematic one. The underlying idea of 'splitting' the reduction process can be found in [40], but in this paper we have further developed some of the results contained there. Throughout this paper we consider several examples and, in some cases, compute the phases that appear in the reconstruction.

Throughout this paper, we work in the category of smooth (i.e. $C^\infty$) objects. For convenience, we will not usually make a notational distinction between a (vector) bundle over a manifold and the ring of its smooth sections, i.e. if $F$ denotes a vector bundle over a manifold $N$ (for instance, a sub-bundle of $TN$), then $X \in F$ simply means that $X : N \to F$ is a section of $F$. The only exception to this rule will be the occasional use of the notation $\mathfrak{X}(N)$ for the ring of smooth vector fields on $N$.

## 2. A general framework for constrained systems

Consider a symplectic manifold $(P, \omega)$, a smooth function $H : P \to \mathbb{R}$ (the Hamiltonian), an embedded submanifold $M$ of $P$ (the constraint submanifold) and a distribution $F$ on $P$ along $M$, i.e. $F$ is a vector sub-bundle of $TP_{|M}$. We are then interested in the following problem: find a smooth section $X$ of the restricted tangent bundle $TP_{|M} \to M$, such that

$$\begin{aligned} &(i_X\omega - \mathrm{d}H)_{|M} \in F^0 \\ &X \in TM \end{aligned} \tag{1}$$

with $F^0$ the annihilator of $F$ in $T^*P_{|M}$. In particular, $X$ then defines a vector field on $M$.

The problem of the existence and uniqueness of the solutions of the constrained system (1) was solved in the following proposition (given in [10]).

**Proposition 2.1.**

*(i) System (1) admits a solution if and only if*

$$\mathrm{d}H_{|M} \in (F \cap TM^\perp)^0.$$

*(ii) If (1) has a solution, then it is unique if and only if*

$$F^\perp \cap TM = 0.$$

Note that the existence condition can be equivalently expressed as

$$X_{H|M} \in TM + F^\perp$$

where $X_H$ denotes the (unconstrained) Hamiltonian vector field on $(P, \omega)$ with Hamiltonian $H$. Hence, any solution $X$ of (1) is of the form

$$X = X_{H|M} + Z \tag{2}$$

for some $Z \in F^\perp$. An interesting special case occurs when rank $F = \dim M$ or, equivalently, $\dim F_x = \dim T_xM$ for all $x \in M$.

**Corollary 2.2 ([10]).** *If* rank $F = \dim M$*, then the condition* $F^\perp \cap TM = 0$ *implies both the existence and uniqueness of a solution of (1).*

Under the conditions of corollary 2.2, (1) is a constrained Hamiltonian system in the sense of Marle [31], who studied such systems in the more general setting of Poisson manifolds.

It is important to point out that if the system admits a solution $X$, it need not be true, in general, that (the restriction of) $H$ is a first integral of $X$. In classical mechanics, for instance, it is well known that imposing nonholonomic constraints on a conservative mechanical system may destroy the conservation of energy (see [31]). An additional assumption on the nature of the constraints is therefore needed to ensure conservation of energy. For a Lagrangian system subject to nonholonomic constraints, a sufficient condition for the energy $E_L$ to be conserved is that the constraints are homogeneous which, in geometrical terms, means that the dilation vector field $\Delta$ should be tangent to the constraint submanifold (see [8, 9, 25]). In the case of linear constraints, this condition is always fulfilled.

*2.1. Nonholonomic Lagrangian systems*

Let us consider a regular Lagrangian system with Lagrangian $L : TQ \to \mathbb{R}$, subject to a set of nonholonomic constraints given by a $(2n - m)$-dimensional submanifold $M$ of $TQ$. $M$ is locally represented by a set of independent functions $\phi_i$, for $1 \leqslant i \leqslant m$: that is, the constraints are merely described by the equations $\phi_i = 0$. For simplicity, in what follows we always assume that $\tau_Q(M) = Q$, i.e. the constraints are 'purely kinematical' in the sense that they do not impose restrictions on the allowable positions. The motions of the system are forced to take place on $M$, and this requires the introduction of some (unknown) 'reaction forces'. In [23, 25], an intrinsic expression for the equations of motion was obtained, which we will describe below.

To fix our notation, let us take $(q^A, \dot{q}^A)$ as the bundle coordinates on $TQ$. We denote by $\Delta = \dot{q}^A \frac{\partial}{\partial \dot{q}^A}$ the dilation vector field on $TQ$ and, by $S = \mathrm{d}q^A \otimes \frac{\partial}{\partial \dot{q}^A}$, the canonical vertical endomorphism (see [27]). Then $\omega_L = -\mathrm{d}S^*(\mathrm{d}L)$ is the Poincaré–Cartan two-form and, $E_L = \Delta L - L$ represents the energy of the system. The symplectic form $\omega_L$ induces two isomorphisms of $C^\infty(TQ)$-modules (*musical mappings*):

$$\flat_L : \mathfrak{X}(TQ) \longrightarrow \Omega^1(TQ) \qquad \sharp_L : \Omega^1(TQ) \longrightarrow \mathfrak{X}(TQ)$$

where $\flat_L(X) = i_X\omega_L$ and $\sharp_L = \flat_L^{-1}$. In the absence of constraints, the dynamics is given by the solution $\Gamma_L$ of the equation $i_{\Gamma_L}\omega_L = \mathrm{d}E_L$, i.e. $\Gamma_L = \sharp_L(\mathrm{d}E_L)$. Indeed, $\Gamma_L$ is a second-order differential equation (SODE) whose solutions are precisely the solutions of the Euler–Lagrange equations for $L$. In the presence of constraints, the equations of motion have to be modified to take them into account.

First of all, we define a distribution $F$ on $TQ$ along $M$ by prescribing its annihilator to be a sub-bundle of $T^*TQ$ which, along the constraint submanifold $M$, represents the bundle of reaction forces. More precisely, we set $F^0 = S^*(TM^0)$. If we write $Z_i = \sharp_L(S^*(\mathrm{d}\phi_i))$, $1 \leqslant i \leqslant m$, we have that $F^\perp$ is locally generated by $Z_1, \ldots, Z_m$.

The equations of motion for the nonholonomic mechanical system are given by

$$\begin{aligned} &(i_X\omega_L - \mathrm{d}E_L)_{|M} \in S^*(TM^0) \\ &X_{|M} \in TM. \end{aligned} \tag{3}$$

It should be pointed out that each solution of (3) (if there exists one) is automatically a SODE along $M$. This implies that, in local coordinates, the integral curves of $X$ on $M$ are of the form $(q^A(t), \dot{q}^A(t))$, whereby $q^A(t)$ are solutions of the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = \lambda^i \frac{\partial \phi_i}{\partial \dot{q}^A} \tag{4}$$

together with the constraint equations $\phi_i(q^A, \dot{q}^A) = 0$, and where $\lambda^i$ are Lagrange multipliers to be determined.

In this case, taking $(P, \omega) = (TQ, \omega_L)$, $H = E_L$, $M$ and $F$ as above, we observe the natural fitting of nonholonomic Lagrangian systems in the model (1). Looking at the problem of existence and uniqueness of solution, we see that the hypothesis of corollary 2.2, rank $F = \dim M$, is fulfilled. So, the nonholonomic system will have a unique solution $X$ if it satisfies the condition $F^\perp \cap TM = 0$ (*the compatibility condition*). If the Hessian of $L$ with respect to the velocities is definite then this condition is automatically satisfied, which is the usual case in mechanics since $L = T - V$, where $T$ is the kinetic energy of a Riemannian metric on $Q$, and $V$ is the potential energy. This will be the assumption made throughout the rest of the paper. If this is the case, a simple dimension count shows that $T_xTQ = F_x^\perp \oplus T_xM$, $\forall x \in M$, which gives rise to two complementary projectors:

$$\mathcal{P}_x : T_xTQ \longrightarrow T_xM \qquad \mathcal{Q}_x : T_xTQ \longrightarrow F_x^\perp.$$

A direct calculation shows that $X = \mathcal{P}(\Gamma_{L|M})$, where $\Gamma_L$ is the solution of the associated unconstrained or free Lagrangian system. In [4, 11, 25], the following alternative approach has been proposed. The compatibility condition is equivalent to the condition that $F \cap TM$ determines a symplectic vector bundle on $M$. Then, $T_xTQ = (F_x \cap T_xM) \oplus (F_x \cap T_xM)^\perp$, $\forall x \in M$, with induced projectors:

$$\bar{\mathcal{P}}_x : T_xTQ \longrightarrow (F_x \cap T_xM) \qquad \bar{\mathcal{Q}}_x : T_xTQ \longrightarrow (F_x \cap T_xM)^\perp.$$

It should be noted that, in general, the projection of the unconstrained dynamics $\Gamma_L$ by $\bar{\mathcal{P}}$ will not produce the constrained dynamics $\Gamma_{L,M}$. However, in the case of homogeneous constraints, we have

$$\bar{\mathcal{P}}(\Gamma_L) = \mathcal{P}(\Gamma_L) = \Gamma_{L,M}$$

along $M$.

## 3. Constrained systems with symmetry: a classification

Let us consider a general constrained system (1) with symmetries. More precisely, let $\Phi$ be a symplectic action $\Phi : G \times P \longrightarrow P$ of a Lie group $G$ on the symplectic manifold $(P, \omega)$, such that the submanifold $M$, the Hamiltonian function $H$ and the vector sub-bundle $F$ are $G$-invariant. For each $g \in G$ and $x \in P$ we put $\Phi(g, x) = \Phi_g(x) = gx$. The infinitesimal generator (fundamental vector field) corresponding to $\xi \in \mathfrak{g}$, with $\mathfrak{g}$ the Lie algebra of $G$, will be denoted by $\xi_P$. The restriction of $\xi_P$ to $M$ is precisely the infinitesimal generator $\xi_M$ of the induced action on $M$.

For simplicity, we will always assume that this action is free and proper. Then, the orbit space $\bar{P} = P/G$ is a differentiable manifold and $\rho : P \longrightarrow \bar{P}$ is a principal bundle with structure group $G$, whereby $\rho$ denotes the natural projection. The $G$-action induced by $\Phi$ on $M$ will still be free and proper. Thus, the quotient manifold $\bar{M} = M/G$ is a smooth submanifold of $\bar{P}$. Finally, the Hamiltonian $H$ will induce a function $\bar{H}$ on $\bar{P}$.

In what follows, we denote by $\mathcal{V}$ the sub-bundle of $TP$, whose fibres are the tangent spaces to the $G$-orbits, i.e. $\mathcal{V}_x = T_x(Gx)$, $\forall x \in P$ or, equivalently, $\mathcal{V} = \ker T\rho$. Note that $\mathcal{V}_x \subset T_xM$ for all $x \in M$, i.e. $\mathcal{V}_{|M} \subset TM$. For simplicity, we will also usually write $\mathcal{V}$, instead of $\mathcal{V}_{|M}$, when referring to its restriction to $M$ (the precise meaning should be clear from the context). If $\xi_M$ is a section of $\mathcal{V} \cap F$, we will call it a *horizontal symmetry* of the given constrained system (see [4, 6]).

We now recall the symplectic reduction established by Cantrijn *et al* in [10], which is just a generalization of the one obtained by Bates and Ṡniatycki for nonholonomic systems (cf [4], see also [18]). So, let us assume that there exists a $G$-invariant solution $X$ of (1) such that $X \in F$. Recall that the latter assumption, in particular, implies that $X(H) = 0$.

**Remark 3.1.** For nonholonomic Lagrangian systems, the condition that the constrained dynamics should belong to the distribution $F$ is not at all restrictive. In fact, from (3), the property that $X \in F$ is a consequence of the fact that $X = \Gamma_{L,M}$ is a SODE.

We define a (generalized) vector sub-bundle $U$ of $TP_{|M}$ by

$$U = (F \cap TM) \cap (\mathcal{V} \cap F)^\perp \tag{5}$$

where $(\mathcal{V} \cap F)^\perp$ is the $\omega$-complement of $\mathcal{V} \cap F$ in $TP_{|M}$. It is not hard to see that $U$ is $G$-invariant and, hence, projects onto a sub-bundle $\bar{U}$ of $T\bar{P}_{|\bar{M}}$. In general, this bundle need not be of constant rank, i.e. it determines a generalized distribution on $P$ along $M$. In the following, however, we will always tacitly assume that $U$ is a genuine vector bundle over $M$.

Let us now denote by $\omega_U$ the restriction of $\omega$ to $U$. Clearly, $\omega_U$ is also $G$-invariant and since, moreover, $i_{\tilde{\xi}}\omega_U = 0$ for all $\tilde{\xi} \in \mathcal{V} \cap U$, the two-form $\omega_U$ pushes down to a two-form $\omega_{\bar{U}}$ on $\bar{U}$ (i.e. $\omega_{\bar{U}}$ only acts on vectors belonging to $\bar{U}$). Similarly, the restriction of $\mathrm{d}H$ to $U$, denoted by $\mathrm{d}_U H$, pushes down to a one-form $\mathrm{d}_{\bar{U}}\bar{H}$ on $\bar{U}$, which is simply the restriction of $\mathrm{d}\bar{H}$ to $\bar{U}$. Note that neither $\omega_{\bar{U}}$ nor $\mathrm{d}_{\bar{U}}\bar{H}$ are differential forms on $\bar{M}$; they are exterior forms on a vector bundle over $\bar{M}$, with smooth dependence on the base point.

**Proposition 3.2 ([10]).** *Let $X$ be a $G$-invariant solution of (1) such that, in addition, $X$ belongs to $F$. Then, the projection $\bar{X}$ of $X$ onto $\bar{M}$ is a section of $\bar{U}$ satisfying the equation*

$$i_{\bar{X}}\omega_{\bar{U}} = \mathrm{d}_{\bar{U}}\,\bar{H}.$$

It is important to observe that, in general, the two-form $\omega_{\bar{U}}$ may be degenerate. However, in the case of a mechanical system with linear nonholonomic constraints, for instance, one can prove that $\omega_{\bar{U}}$ is nondegenerate, such that $(\bar{U}, \omega_{\bar{U}})$ becomes a symplectic vector bundle over $\bar{M}$ (see [4]). The reduced dynamics is then uniquely determined by the equation mentioned in proposition 3.2.

Next, following [10], we will identify three classes of constrained systems with symmetry. This classification arises from carefully considering the intersection $\mathcal{V} \cap F$, which points out how well the symmetries fit in the constrained system.

(i) *The general case:* $\{0\} \subsetneq \mathcal{V}_x \cap F_x \subsetneq \mathcal{V}_x$, for all $x \in M$.

(ii) *The purely kinematic case:* $\mathcal{V}_x \cap F_x = \{0\}$ and $T_xM = \mathcal{V}_x + (F_x \cap T_xM)$, for all $x \in M$.

(iii) *The case of horizontal symmetries:* $\mathcal{V}_x \cap F_x = \mathcal{V}_x$, for all $x \in M$, which is equivalent to $\mathcal{V}_x \subset F_x$, for all $x \in M$.

## 4. The general case

Consider the case where at each $x \in M$, $\{0\} \subsetneq \mathcal{V}_x \cap F_x \subsetneq \mathcal{V}_x$. Let us assume that the given action of $G$ on $P$ is Hamiltonian, with momentum mapping $J$. If we make the corresponding computations, we see that $J$ is no longer a conserved quantity for the constrained dynamics. However, extending a procedure developed by Bloch *et al* [6] for nonholonomic mechanical systems (see also [7]), Cantrijn *et al* [10] derived an equation which describes the evolution of some components of the momentum mapping along the integral curves of the constrained system. Which components? Just those which come from the symmetries which are compatible with the bundle $F$.

More precisely, for each $x \in M$, we put

$$\mathfrak{g}^x = \{\xi \in \mathfrak{g} | \xi_M(x) \in F_x\}.$$

Recall that $\xi_M$ is just the restriction of $\xi_P$ to the $G$-invariant submanifold $M$. We have that $\mathfrak{g}^x$ is a vector subspace of $\mathfrak{g}$. Putting

$$\mathfrak{g}^F = \coprod_{x \in M} \mathfrak{g}^x$$

where we use the symbol '$\coprod$' to denote the disjoint union of the vector spaces, we obtain a ('generalized') vector bundle over $M$, with canonical projection $\mathfrak{g}^F \to M : \xi \in \mathfrak{g}^x \mapsto x$. In general, this bundle need not have constant rank. However, for the subsequent discussion we make the simplifying assumption that $\mathfrak{g}^F$ is a genuine vector bundle over $M$, the fibres of which have constant dimension (independent of the base point).

Suppose now that the symplectic form $\omega$ is exact, say $\omega = \mathrm{d}\theta$, and that the $G$-action leaves $\theta$ invariant. In such a case there always exists a well-defined momentum mapping $J : P \longrightarrow \mathfrak{g}^*$ such that

$$\langle J(x), \xi\rangle = -(\theta_x)(\xi_P(x)) \qquad \forall x \in P \qquad \forall \xi \in \mathfrak{g}$$

(see [1]). Herewith, we can define a smooth section $J^{(c)} : M \longrightarrow (\mathfrak{g}^F)^*$ of the dual bundle $(\mathfrak{g}^F)^*$ as follows:

$$J^{(c)}(x) : \mathfrak{g}^x \longrightarrow \mathbb{R} \qquad J^{(c)}(x)(\xi) = \langle J(x), \xi\rangle.$$

$J^{(c)}$ will be called the *constrained momentum mapping* [6, 7, 10]. Given a smooth section $\bar{\xi}$ of the vector bundle $\mathfrak{g}^F$, we can then define a smooth function $J^{(c)}_{\bar{\xi}}$ on $M$ according to

$$J^{(c)}_{\bar{\xi}} = \langle J^{(c)}, \bar{\xi}\rangle.$$

In addition, we can construct a vector field $\Xi$ on $M$ by putting

$$\Xi(x) = (\bar{\xi}(x))_M(x) \qquad \forall x \in M.$$

Denoting the Lie derivative operator with respect to $\Xi$ as $\mathcal{L}_\Xi$ leads to the following theorem.

**Theorem 4.1 ([10]).** *Let $X$ be an arbitrary solution of (1). For any smooth section $\bar{\xi}$ of $\mathfrak{g}^F$ we then have*

$$X(J^{(c)}_{\bar{\xi}}) = -(\mathcal{L}_\Xi\theta)(X). \tag{6}$$

Note that for the above result we do not have to enforce the requirement that $X$ is $G$-invariant. Equation (6) is called *the momentum equation* for the given constrained system. In the case of linear nonholonomic constraints we precisely recover the result established by Bloch *et al* [6].

**Remark 4.2.** Suppose again that $X$ is a solution of (1) and let $\bar{\xi}$ be a constant section of $\mathfrak{g}^F$, i.e. $\bar{\xi}(x) = \xi^0 \in \mathfrak{g}$ for all $x \in M$. We may then identify the corresponding vector field $\Xi$ with the infinitesimal generator $\xi^0_M$ and, clearly, $J^{(c)}_{\bar{\xi}} = (J_{\xi^0})_{|M}$. Moreover, by construction, $\xi^0_M$ is a horizontal symmetry. The momentum equation (6) then leads to

$$X(J^{(c)}_{\bar{\xi}}) = X(J_{\xi^0})_{|M} = 0$$

i.e., we have obtained a conserved quantity of $X$ associated with the horizontal symmetry $\xi^0_M$. This is a manifestation of Noether's theorem for constrained systems.

### *4.1. Reduction*

In this section, we are going to perform reduction in the general case via the constrained momentum mapping. To fix ideas, *we will work specifically with nonholonomic Lagrangian systems*.

**Remark 4.3.** We note that for nonholonomic Lagrangian systems (see [6, 9]) the reduction theory is developed in terms of the vector bundle $\mathfrak{g}^M \longrightarrow Q$, defined by

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} | \xi_{TQ}(v_q) \in F_{v_q} \text{ for all } v_q \in M \cap T_qQ\}.$$

The nonholonomic momentum mapping $J^{nh} : TQ \longrightarrow (\mathfrak{g}^M)^*$ is then defined by

$$\langle J^{nh}(v_q), \xi\rangle = \alpha_L(\xi_{TQ})(v_q)$$

for all $v_q \in TQ$ and $\xi \in \mathfrak{g}^q$. In fact, $J^{nh}$ restricts naturally to $M$, $J^{nh}_{|M} : M \longrightarrow (\mathfrak{g}^M)^*$. For simplicity, we will usually denote this mapping by $J^{nh}$, instead of $J^{nh}_{|M}$. We will only recover the distinction when confusion is possible.

A global section $\bar{\xi}$ of the vector bundle $\mathfrak{g}^M \longrightarrow Q$ induces a vector field $\Xi$ on $Q$ as follows:

$$\Xi(q) = (\bar{\xi}(q))_Q(q) \in T_q Q$$

for all $q \in Q$. Then the nonholonomic momentum equation reads as

$$\Gamma_{L,M}(J^{nh}_{\bar{\xi}}) = \Xi^c(L).$$

We will investigate the relation between the vector bundles $\mathfrak{g}^M \longrightarrow Q$ and $\mathfrak{g}^F \longrightarrow M$ defined above. By definition, we have that

$$\mathfrak{g}^q = \bigcap_{v_q \in M \cap T_q Q} \mathfrak{g}^{v_q}$$

for all $q \in Q$. However, the fibres do not generally coincide. Indeed, let us take $\xi \in \mathfrak{g}^{v_q}$ and $w_q \in M \cap T_q Q$. We want to see if $\xi \in \mathfrak{g}^{w_q}$, i.e. $\xi_{TQ}(w_q) \in F_{w_q}$. Applying the musical mapping $\flat_L$, this is equivalent to $\flat_L(\xi_{TQ}(w_q)) = (\mathrm{d}J_\xi)_{w_q} \in \flat_L(F_{w_q}) = (F^\perp_{w_q})^0$. Now, $F^\perp$ is locally generated by the Hamiltonian vector fields $Z_1, \ldots, Z_m$ (cf section 2.1). Consequently, we would have

$$(\mathrm{d}J_\xi)_{w_q} Z_i(w_q) = 0 \qquad 1 \leqslant i \leqslant m.$$

But $(\mathrm{d}J_\xi)_{w_q} Z_i(w_q) = \omega_L(\xi_{TQ}, Z_i)(w_q) = -S^*(\mathrm{d}\phi_i)_{w_q}(\xi_{TQ}(w_q)) = -(\mathrm{d}\phi_i)_{w_q}(\xi^v_Q(w_q))$. In coordinates, if we write $\xi_Q(q) = f^A(q)\frac{\partial}{\partial q^A}$, this reads as

$$(\mathrm{d}\phi_i)_{w_q}(\xi^v_Q(w_q)) = \frac{\partial \phi_i}{\partial \dot{q}^A}(q, \dot{q}) f^A(q).$$

Then, if the constraints are linear or affine, $\frac{\partial \phi_i}{\partial \dot{q}^A}(q, \dot{q})$ only depends on the base point $q \in Q$, and $\xi \in \mathfrak{g}^{v_q}$ implies $\xi \in \mathfrak{g}^{w_q}$ for all $w_q \in M \cap T_q Q$. Therefore, $\mathfrak{g}^q = \mathfrak{g}^{w_q}$ for all $w_q \in M \cap T_q Q$.

As we have remarked above, the main difficulty (and just the point) for nonholonomic systems is that the momenta is not a conserved quantity. So, instead of fixing a value $\mu \in \mathfrak{g}^*$ for the momentum as in the traditional approach of symplectic reduction [1, 36, 37], we will take a $C^\infty$-section $\mu : Q \longrightarrow (\mathfrak{g}^M)^*$ of the dual vector bundle $(\mathfrak{g}^M)^*$, with canonical projection $\pi^* : (\mathfrak{g}^M)^* \longrightarrow Q$, which gives the momenta along the integral curves of the dynamics $\Gamma_{L,M}$. Now, consider the level set

$$(J^{nh})^{-1}(\mu) = \{v_q \in M | J^{nh}(v_q) = \mu(q)\}.$$

In general, $(J^{nh})^{-1}(\mu)$ will not be a submanifold of $M$. We will denote the inclusion by $j : (J^{nh})^{-1}(\mu) \hookrightarrow M$.

Assume that the vector bundle $\mathfrak{g}^M \longrightarrow Q$ has constant rank $r$, and choose $\bar{\xi}_1, \ldots, \bar{\xi}_r$, $r$ linearly independent sections. Consider $r$ functions on $M$, $f_i : M \longrightarrow \mathbb{R}$, defined by $f_i = \langle \mu, \bar{\xi}_i \rangle \circ \tau_Q - J^{nh}_{\bar{\xi}_i}$. For each $i$, we denote $P_{\bar{\xi}_i} = f_i^{-1}(0)$. Then, it is not difficult to see that

$$(J^{nh})^{-1}(\mu) = \bigcap_{i=1}^{r} P_{\bar{\xi}_i}.$$

In the following proposition, we characterize the section $\mu$ and give certain conditions to assure the existence of a differentiable structure on $(J^{nh})^{-1}(\mu)$.

**Proposition 4.4.** *If 0 is a weakly regular value of $f_i$ for $1 \leqslant i \leqslant r$, then $P_{\bar{\xi}_i}$ is a submanifold of $M$. If, in addition, the intersection $\bigcap_{i=1}^r P_{\bar{\xi}_i}$ is clean, then $(J^{nh})^{-1}(\mu)$ is a submanifold of $M$, and $\Gamma_{L,M}$ is at a tangent to it if and only if*

$$\Gamma_{L,M}(\langle \mu, \bar{\xi}_i \rangle \circ \tau_Q) = \Xi^c_i(L) \tag{7}$$

*for all $1 \leqslant i \leqslant r$.*

**Proof.** Because of the above discussion, it only remains to prove the equivalence. Assume that the section $\mu$ fulfils condition (7), namely, $\Gamma_{L,M}(\langle \mu, \bar{\xi}_i\rangle \circ \tau_Q - J^{nh}_{\bar{\xi}_i}) = \Xi^c_i(L)$, for all $1 \leqslant i \leqslant r$. Then, due to the nonholonomic momentum equation, we just have that $\Gamma_{L,M}(f_i) = 0$. So $\Gamma_{L,M}$ is tangent to the level submanifold $P_{\bar{\xi}_i}$ for each $i$. As $T(J^{nh})^{-1}(\mu) = \bigcap_{i=1}^r TP_{\bar{\xi}_i}$, it follows that $\Gamma_{L,M} \in T(J^{nh})^{-1}(\mu)$. The converse is obvious. □

**Remark 4.5.** The hypothesis of 0 to be a weakly regular value of $f_i$, for $1 \leqslant i \leqslant r$, is the natural translation of fixing a weakly regular value of the momentum in the approach by Marsden and Weinstein [36].

In the following, we will assume the hypothesis of proposition 4.4.

**Lemma 4.6.** *We have*

$$T^{\perp}(J^{nh}_{|M})^{-1}(\mu) = T^{\perp}M + \langle X_{\tilde{f}_1}, \ldots, X_{\tilde{f}_r}\rangle.$$

**Proof.** We will now distinguish between $J^{nh} : TQ \longrightarrow (\mathfrak{g}^M)^*$ and $J^{nh}_{|M} : M \longrightarrow (\mathfrak{g}^M)^*$. We have that $(J^{nh}_{|M})^{-1}(\mu) = (J^{nh})^{-1}(\mu) \cap M$. Consider $\tilde{f}_i$, the natural extension of $f_i$ to $TQ$, $\tilde{f}_i = \langle \mu, \bar{\xi}_i\rangle \circ \tau_Q - J^{nh}_{\bar{\xi}_i}$. For each $i$, denote $\tilde{P}_{\bar{\xi}_i} = \tilde{f}_i^{-1}(0)$. It is clear that $P_{\bar{\xi}_i} = \tilde{P}_{\bar{\xi}_i} \cap M$. We also have that $(J^{nh})^{-1}(\mu) = \bigcap_{i=1}^r \tilde{P}_{\bar{\xi}_i}$. A simple dimension count shows that $\dim T(J^{nh})^{-1}(\mu) \geqslant \dim TQ - r$. Consequently, we have that $\dim T^{\perp}(J^{nh})^{-1}(\mu) \leqslant r$. On the other hand, it easy to check that $X_{\tilde{f}_i} \in T^{\perp}(J^{nh})^{-1}(\mu), 1 \leqslant i \leqslant r$. Then, we have proved that $T^{\perp}(J^{nh})^{-1}(\mu) = \langle X_{\tilde{f}_1}, \ldots, X_{\tilde{f}_r}\rangle$. Finally, $T^{\perp}(J^{nh}_{|M})^{-1}(\mu) = T^{\perp}M + T^{\perp}(J^{nh})^{-1}(\mu) = T^{\perp}M + \langle X_{\tilde{f}_1}, \ldots, X_{\tilde{f}_r}\rangle$. □

In order to perform reduction, we need a kind of $G$-action on the vector bundle $(\mathfrak{g}^M)^*$, playing the role of the coadjoint action of $G$ on $\mathfrak{g}^*$. The following lemma enables us to go further in that direction.

**Lemma 4.7.** *Let $Ad^M : G \times \mathfrak{g}^M \longrightarrow \mathfrak{g}^M$ be defined by $Ad^M(g, \xi) = Ad_g(\xi)$, for each $g \in G$ and $\xi \in \mathfrak{g}^q$. Then $Ad^M$ is a well-defined 'action' on the vector bundle $\mathfrak{g}^M$.*

**Proof.** The unique fact to be proved is that $Ad^M$ is well defined, because the properties $Ad^M_e = Id$ and $Ad^M_{gh} = Ad^M_g \circ Ad^M_h$ follows directly from the fact that $Ad$ is a $G$-action. Thus let us take $g \in G$ and $\xi \in \mathfrak{g}^q$, which is to say that $\xi_{TQ}(v_q) \in F_{v_q}$ for all $v_q \in T_qQ \cap M$. As the vector bundle $F$ is $G$-invariant, we have that $(Ad_g(\xi))_{TQ}(g \cdot v_q) = (\Phi_g)_*(\xi_{TQ}(v_q))$ belongs to $F_{g \cdot v_q}$, for all $v_q \in T_qQ \cap M$, namely, $(Ad_g(\xi))_{TQ}(w_q) \in F_{w_q}$, for all $w_q \in T_qQ \cap M$. Consequently, $Ad_g(\xi) \in \mathfrak{g}^{gq}$ and $Ad^M$ is well defined. □

In a similar way, we can consider the $G$-'action' on $(\mathfrak{g}^M)^*$ defined by

$$\begin{aligned} CoAd^M : G \times (\mathfrak{g}^M)^* &\longrightarrow (\mathfrak{g}^M)^* \\ (g, \eta) &\longmapsto CoAd^M(g, \eta) = CoAd_g(\eta). \end{aligned}$$

Note that the nonholonomic momentum mapping $J^{nh} : M \longrightarrow (\mathfrak{g}^M)^*$ is $G$-equivariant, that is, the following diagram:

$$\begin{array}{cccccc} & M & \xrightarrow{J^{nh}} & (\mathfrak{g}^M)^* & \\ \Phi_g & \downarrow & & \downarrow & CoAd^M_g \\ & M & \xrightarrow{J^{nh}} & (\mathfrak{g}^M)^* & \end{array}$$

is commutative: $CoAd_g(J^{nh}(v_q)) = J^{nh}(g \cdot v_q)$, for all $g \in G$.

**Remark 4.8.** The concept of $G$-equivariance can be defined for general constrained dynamical systems with symmetry in a similar way. It is clear that if the Hamiltonian action of the Lie group has a $G$-equivariant momentum mapping $J$, the corresponding constrained momentum mapping $J^{(c)}$ will also be equivariant.

The last ingredient we need to define is the 'isotropy group' of the action $CoAd^M$ corresponding to the section $\mu : Q \longrightarrow (\mathfrak{g}^M)^*$. This is defined as

$$G_\mu = \{g \in G | CoAd^M(\mu) = \mu\}$$

where we mean by $CoAd^M(\mu) = \mu$ that $CoAd^M_g(\mu(q)) = \mu(gq)$ for all $q \in Q$. It is not difficult to see that $G_\mu$ is a Lie subgroup of $G$.

Therefore, we can define a $G_\mu$-action on the manifold $(J^{nh})^{-1}(\mu)$ in the following manner:

$$\begin{aligned} \Theta : G_\mu \times (J^{nh})^{-1}(\mu) &\longrightarrow (J^{nh})^{-1}(\mu) \\ (g, v_q) &\longmapsto \Theta(g, v_q) = g \cdot v_q. \end{aligned}$$

The definition of the group $G_\mu$ and the equivariance of $J^{nh} : M \longrightarrow (\mathfrak{g}^M)^*$ implies that this action is well defined, as we propose in the following lemma.

**Lemma 4.9.** *The mapping $\Theta$ is well defined.*

**Proof.** Take $g \in G_\mu$ and $v_q \in (J^{nh})^{-1}(\mu)$. By the equivariance, we have that $J^{nh}(\Theta(g, v_q)) = CoAd^M_g(J^{nh}(v_q)) = CoAd^M_g(\mu(q))$. Finally, by the definition of $G_\mu$, it follows that $\Theta(g, v_q) \in (J^{nh})^{-1}(\mu)$. □

We can consider the action $\Theta$ as the restriction to $(J^{nh})^{-1}(\mu)$ of a $G_\mu$-action on $M$, $\Theta_M : G_\mu \times M \longrightarrow M$. Both $\Theta$ and $\Theta_M$ will be free and proper actions, because they inherite these properties from the original action $\Phi : G \times TQ \longrightarrow TQ$. Then, the orbit spaces $M/G_\mu$ and $\overline{(J^{nh})^{-1}(\mu)} = (J^{nh})^{-1}(\mu)/G_\mu$ are differentiable manifolds, and we have two principal $G_\mu$-bundles $\pi : M \longrightarrow M/G_\mu$ and $\pi_{|(J^{nh})^{-1}(\mu)} : (J^{nh})^{-1}(\mu) \longrightarrow \overline{(J^{nh})^{-1}(\mu)}$, respectively.

*4.1.1. A kind of symplectic reduction.* Now, we define a (generalized) vector sub-bundle $U_\mu$ of $TM_{|(J^{nh})^{-1}(\mu)}$, whose fibre at $x \in (J^{nh})^{-1}(\mu)$ is given by

$$(U_\mu)_x = \{v \in F_x \cap T_x(J^{nh})^{-1}(\mu) / \omega_L(v, \tilde{\xi}) = 0, \text{ for all } \tilde{\xi} \in (\mathcal{V}_\mu)_x \cap F_x\}. \tag{8}$$

In general, $U_\mu$ need not be of constant rank. For the further discussion, however, we will assume that $U_\mu$ is a genuine vector bundle over $(J^{nh})^{-1}(\mu)$. Note that $U = F \cap T(J^{nh})^{-1}(\mu) \cap (\mathcal{V}_\mu \cap F)^\perp$, where $(\mathcal{V}_\mu \cap F)^\perp$ is the $\omega_L$-complement of $\mathcal{V}_\mu \cap F$ in $TTQ_{|(J^{nh})^{-1}(\mu)}$. $U_\mu$ is $G_\mu$-invariant and, hence, it projects onto a sub-bundle $\bar{U}_\mu$ of $T(\bar{M}_\mu)_{|\overline{(J^{nh})^{-1}(\mu)}}$.

Let us now denote by $\omega_\mu$ the restriction of $\omega_L$ to $U_\mu$. Clearly, $\omega_\mu$ is also $G_\mu$-invariant and by the very definition of the vector bundle $U_\mu$, the two-form $\omega_\mu$ pushes down to a two-form $\bar{\omega}_\mu$ on $\bar{U}_\mu$. Similarly, the restriction of $\mathrm{d}E_L$ to $U_\mu$, denoted by $\mathrm{d}_\mu E_L$, pushes down to a one-form $\mathrm{d}_\mu \bar{E}_L$ on $\bar{U}_\mu$, which is simply the restriction of $\mathrm{d}\bar{E}_L$ to $\bar{U}_\mu$. Note that neither $\bar{\omega}_\mu$ nor $\mathrm{d}_\mu \bar{E}_L$ are differential forms on $\overline{(J^{nh})^{-1}(\mu)}$; they are exterior forms on a vector bundle over $\overline{(J^{nh})^{-1}(\mu)}$, with smooth dependence on the base point.

**Proposition 4.10.** *Let $\Gamma_{L,M}$ be the solution of (3). Then, its projection $(\bar{\Gamma}_{L,M})_\mu$ onto $\overline{(J^{nh})^{-1}(\mu)}$ is a section of $\bar{U}_\mu$ satisfying the equation*

$$i_{(\bar{\Gamma}_{L,M})_\mu} \bar{\omega}_\mu = \mathrm{d}_\mu \bar{E}_L.$$

**Proof.** Similar to proposition 3.2. □

**Remark 4.11.** It should be noticed that, in general, the two-form $\bar{\omega}_\mu$ may be degenerate. So, the reduced dynamics is not uniquely determined by the equation mentioned in proposition 4.10.

*4.1.2. Almost-Poisson reduction.* For nonholonomic Lagrangian systems, we know [8, 9, 19, 32, 42] that on $M$ the so-called nonholonomic bracket, $\{\cdot,\cdot\}_M$, can be constructed in the following manner. Consider $\lambda, \sigma : M \longrightarrow \mathbb{R}$ and take $\tilde{\lambda}, \tilde{\sigma}$ arbitrary extensions to $TQ$, $\tilde{\lambda} \circ j_M = \lambda$, $\tilde{\sigma} \circ j_M = \sigma$, with $j_M : M \hookrightarrow TQ$. Then

$$\{\lambda, \sigma\}_M = \omega_L(\bar{\mathcal{P}}(X_{\tilde{\lambda}}), \bar{\mathcal{P}}(X_{\tilde{\sigma}})) \circ j_M.$$

It is a routine procedure to verify that this bracket is well defined. In general, $\{\cdot,\cdot\}_M$ does not verify the Jacobi identity, except if the constraints are holonomic. This almost-Poisson bracket is very important because, in the case of homogeneous constraints, it gives the evolution of the constrained dynamics in the following sense: for any function $f \in C^\infty(M)$, its evolution along integral curves of $\Gamma_{L,M}$ on $M$ is given by

$$\dot{f} = \Gamma_{L,M}(f) = \{f, E_L\}_M.$$

The idea of this approach is to project the nonholonomic bracket onto the reduced space $\overline{(J^{nh})^{-1}(\mu)}$ via the $G_\mu$-action $\Theta : G_\mu \times (J^{nh})^{-1}(\mu) \longrightarrow (J^{nh})^{-1}(\mu)$. For this purpose we briefly recall the main results of the Poisson reduction stated in [35, 41], but from an almost-Poisson point of view.

**Definition 4.12.** *Let $(M, \Lambda_M)$ be an almost-Poisson manifold. Then a pair $(N, E)$ that consists of a submanifold $j : N \subseteq M$, and a vector sub-bundle $E$ of $TM_{|N}$ will be called a reductive structure of $(M, \Lambda_M)$ if the following conditions are satisfied:*

*(i) $E \cap TN$ is tangent to a foliation $\mathcal{F}$ whose leaves are the fibres of a submersion $\pi : N \longrightarrow S$;*
*(ii) For all $\varphi, \psi \in C^\infty(M)$ such that $\mathrm{d}\varphi$ and $\mathrm{d}\psi$ vanish on $E$, $\mathrm{d}\{\varphi, \psi\}_M$ also vanishes on $E$. Furthermore, if $S$ above has an almost-Poisson structure $\Lambda_S$ such that for any local $C^\infty$ functions $f, g$ on $S$, and any local extensions $\varphi, \psi$ of $\pi^* f, \pi^* g$, with $\mathrm{d}\varphi_{|E} = \mathrm{d}\psi_{|E} = 0$, the relation*

$$\{\varphi, \psi\}_M \circ j = \{f, g\}_S \circ \pi$$

*holds true, we say that $(M, N, E)$ is a reducible triple, and $(S, \Lambda_S)$ is the reduced almost-Poisson manifold of $(M, \Lambda_M)$ via $(N, E)$.*

The bundle $E$ is sometimes called the control bundle. The following theorem characterizes the reducible triples.

**Theorem 4.13.** *Let $(N, E)$ a reductive structure of the almost-Poisson manifold $(M, \Lambda_M)$. Then $(M, N, E)$ is a reducible triple iff*

$$\sharp_M(E^0) \subseteq TN + E.$$

So, in our case, we have that $N = (J^{nh})^{-1}(\mu)$. It seems to be quite reasonable to take as $E$, at each point $v_q$ of $(J^{nh})^{-1}(\mu)$, just the tangent at $v_q$ to the $G_\mu$-orbit of $v_q$, i.e. $E_{v_q} = T_{v_q}(G_\mu \cdot v_q)$. It is to easy see that $(N, E)$ is a reductive structure, with $S = \overline{(J^{nh})^{-1}(\mu)}$. We will discuss if $(M, N, E)$ is a reducible triple. We have that

$$E^0 = \langle \mathrm{d}\chi / \chi \in C^\infty_{G_\mu}(M) \rangle$$

where $C^{\infty}_{G_\mu}(M)$ denotes the $G_\mu$-invariant functions on $M$. Then,

$$\sharp_M(E^0) = \langle X^M_\chi / \chi \in C^{\infty}_{G_\mu}(M)\rangle.$$

Note that $X^M_\chi$ denotes the Hamiltonian vector field associated with the function $\chi : M \longrightarrow \mathbb{R}$ by the musical mapping $\sharp_M$ induced by the almost-Poisson bivector field $\Lambda_M$. But,

$$\begin{aligned} X^M_\chi(\upsilon) &= \{\upsilon, \chi\}_M = \omega_L(\bar{\mathcal{P}}(X_{\tilde{\upsilon}}), \bar{\mathcal{P}}(X_{\tilde{\chi}})) \circ j_M \\ &= \omega_L(X_{\tilde{\upsilon}}, \bar{\mathcal{P}}(X_{\tilde{\chi}})) \circ j_M = \bar{\mathcal{P}}(X_{\tilde{\chi}})(\upsilon) \end{aligned}$$

for all $\upsilon \in C^\infty(M)$, where $\tilde{\chi}$ denotes an arbitrary extension of $\chi$ to $TQ$. So $X^M_\chi = \bar{\mathcal{P}}(X_{\tilde{\chi}})$ and

$$\sharp_M(E^0) = \langle \bar{\mathcal{P}}(X_{\tilde{\chi}}) / \chi \in C^{\infty}_{G_\mu}(M)\rangle.$$

In addition, $E + T(J^{nh})^{-1}(\mu) = T(J^{nh})^{-1}(\mu)$, then we have

$$\begin{aligned} \sharp_M(E^0) \subseteq T(J^{nh})^{-1}(\mu) &\Longleftrightarrow \bar{\mathcal{P}}(X_{\tilde{\chi}})(f_i) \circ j = 0 \qquad 1 \leqslant i \leqslant r \qquad \forall \chi \in C^{\infty}_{G_\mu}(M) \\ &\Longleftrightarrow \{f_i, \chi\}_M \circ j = 0 \qquad 1 \leqslant i \leqslant r \qquad \forall \chi \in C^{\infty}_{G_\mu}(M). \end{aligned} \tag{9}$$

In the purely kinematic case, as we will discuss below, the nonholonomic momentum mapping is trivial, and therefore conditions (9) hold trivially (in fact, $(J^{nh})^{-1}(\mu) = M$). In the horizontal case, we would have $\mathfrak{g}^M = \mathfrak{g} \times Q$, so $r = \dim G$. Taking a constant section $\mu(q) = (\mu, q)$ and a basis of the Lie algebra $\mathfrak{g}$, $\xi_1, \ldots, \xi_r$, we could write $f_i = \langle \mu, \xi_i\rangle - J_{\xi_i}$, $1 \leqslant i \leqslant r$. Then $\{f_i, \chi\}_M \circ j = -\bar{\mathcal{P}}(X_{f_i})(\chi) \circ j = (\xi_i)_M(\chi) \circ j$. In general, conditions (9) will not be fulfilled, because $C^{\infty}_{G_\mu}(M) \neq C^{\infty}_{G}(M)$.

*4.1.3. Almost-Poisson mappings.* The obstruction we have found above in the horizontal case to reduce the nonholonomic bracket $\{\cdot,\cdot\}_M$ to $\overline{(J^{nh})^{-1}(\mu)}$ via $((J^{nh})^{-1}(\mu)), T(G_\mu \cdot))$ leads us to develop another reduction scheme which takes into account the whole group $G$. For that purpose, let us define the following mapping:

$$k : \overline{(J^{nh})^{-1}(\mu)} \xrightarrow{k_\mu} M/G_\mu \xrightarrow{p} M/G = \bar{M}.$$

On $\bar{M}$, we have the natural almost-Poisson structure induced by $(M, \Lambda_M)$. The idea of this section is to study under which conditions there exists an almost-Poisson structure on $\overline{(J^{nh})^{-1}(\mu)}$ so that $k$ is an almost-Poisson mapping. In this case, then for each pair of functions $\bar{\lambda}, \bar{\sigma} : \bar{M} \longrightarrow \mathbb{R}$, we would have that

$$\{\lambda_\mu, \sigma_\mu\}_\mu = \{\bar{\lambda}, \bar{\sigma}\}_{\bar{M}} \circ k$$

with $\bar{\lambda} \circ k = \lambda_\mu$ and $\bar{\sigma} \circ k = \sigma_\mu$.

In fact, taking $\bar{\lambda}_1, \bar{\lambda}_2 : M \longrightarrow \mathbb{R}$ with $\bar{\lambda}_1 \circ k = \bar{\lambda}_2 \circ k = \lambda_\mu$, we would have

$$\{\bar{\lambda}_1, \bar{\sigma}\}_{\bar{M}} \circ k = \{\bar{\lambda}_2, \bar{\sigma}\}_{\bar{M}} \circ k \qquad \forall \bar{\sigma} \in C^\infty(\bar{M}). \tag{10}$$

In case of $k$ being injective, this equality would be a necessary and sufficient condition to obtain an almost-Poisson bracket $\{\cdot,\cdot\}_\mu$ on $\overline{(J^{nh})^{-1}(\mu)}$, making $k$ an almost-Poisson morphism. Moreover, in this case, $\{\cdot,\cdot\}_\mu$ will uniquely satisfy that property.

We will discuss if equality (10) is fulfilled. Equivalently, given $\bar{\lambda} : \bar{M} \longrightarrow \mathbb{R}$ with $\bar{\lambda} \circ k = 0$, we want to verify if

$$\{\bar{\lambda}, \bar{\sigma}\}_{\bar{M}} \circ k = 0 \qquad \forall \bar{\sigma} \in C^\infty(\bar{M}).$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc} & (J^{nh})^{-1}(\mu) & \xrightarrow{j} & M & \longrightarrow & M & \\ \pi_{(J^{nh})^{-1}(\mu)} & \downarrow & & \downarrow & & \downarrow & \rho_{|M} \\ & \overline{(J^{nh})^{-1}(\mu)} & \xrightarrow{k_\mu} & M/G_\mu & \xrightarrow{p} & \bar{M} & . \end{array}$$

Then, we have that $\{\rho^*_{|M}\bar\lambda, \rho^*_{|M}\bar\sigma\}_M \circ j = \{\bar\lambda, \bar\sigma\}_{\bar M} \circ \rho_{|M} \circ j = \{\bar\lambda, \bar\sigma\}_{\bar M} \circ k \circ \pi_{(J^{nh})^{-1}(\mu)}$. In addition, $\rho^*_{|M}\bar\lambda \circ j = \bar\lambda \circ k \circ \pi_{(J^{nh})^{-1}(\mu)} = 0$.

It is clear that

$$\{\rho^*_{|M}\bar\lambda, \rho^*_{|M}\bar\sigma\}_M \circ j = 0 \Longleftrightarrow \{\bar\lambda, \bar\sigma\}_{\bar M} \circ k = 0.$$

Therefore, our question can now be presented as follows: given $\lambda \in C^\infty_G(M)$ with $\lambda \circ j = 0$, we want to verify if

$$\{\lambda, \sigma\}_M \circ j = 0 \qquad \forall \sigma \in C^\infty_G(M).$$

By definition, we have that $\{\lambda, \sigma\}_M = \omega_L(\bar{\mathcal{P}}(X_{\tilde\lambda}), \bar{\mathcal{P}}(X_{\tilde\sigma})) \circ j_M$, where $\tilde\lambda$, $\tilde\sigma$ are arbitrary extensions of $\lambda, \sigma$ to $TQ$, $\tilde\lambda \circ j_M = \lambda$, $\tilde\sigma \circ j_M = \sigma$. Without loss of generality, we can suppose them to be $G$-invariant.

Now, $(j_M \circ j)^*\tilde\lambda = j^*\lambda = 0$. Therefore, we deduce

$$0 = (j_M \circ j)^* \mathrm{d}\tilde\lambda = (j_M \circ j)^* i_{X_{\tilde\lambda}}\omega_L.$$

If we could assure that $\bar{\mathcal{P}}(X_{\tilde\sigma}) \in T(J^{nh})^{-1}(\mu)$, then we would have

$$\begin{aligned}\{\lambda, \sigma\}_M \circ j &= \omega_L(X_{\tilde\lambda}, \bar{\mathcal{P}}(X_{\tilde\sigma})) \circ (j_M \circ j)\\ &= \omega_L(X_{\tilde\lambda}, (j_M \circ j)_*\bar{\mathcal{P}}(X_{\tilde\sigma})) \circ (j_M \circ j)\\ &= (j_M \circ j)^* i_{X_{\tilde\lambda}}\omega_L(\bar{\mathcal{P}}(X_{\tilde\sigma})\\ &= 0.\end{aligned}$$

Therefore, if we guarantee that $\bar{\mathcal{P}}(X_{\tilde\sigma}) \in T(J^{nh})^{-1}(\mu)$, $\forall\tilde\sigma \in C^\infty_G(TQ)$, then (10) holds. We characterize when this occurs in the following proposition.

**Proposition 4.14.** *Let $\sigma$ be a $G$-invariant function on $M$, and $\tilde\sigma$ one $G$-invariant extension of $\sigma$ to TQ. Then,*

$$\bar{\mathcal{P}}(X_{\tilde\sigma}) \in T(J^{nh})^{-1}(\mu) \Longleftrightarrow \{\sigma, f_i\}_M \circ j = 0 \qquad 1 \leqslant i \leqslant r. \tag{11}$$

**Proof.** Take $\sigma \in C^\infty_G(M)$. We have that

$$\bar{\mathcal{P}}(X_{\tilde\sigma}) \in T(J^{nh})^{-1}(\mu) \Longleftrightarrow \omega_L(\bar{\mathcal{P}}(X_{\tilde\sigma}), Z) \circ j_M \circ j = 0 \qquad \forall Z \in T^\perp(J^{nh})^{-1}(\mu).$$

By lemma 4.6, we know that $T^\perp(J^{nh})^{-1}(\mu) = T^\perp M + \langle X_{\tilde f_1}, \ldots, X_{\tilde f_r}\rangle$. As $F \cap TM \subset TM$, then $T^\perp M \subset (F \cap TM)^\perp$.Thus we have that $\omega_L(\bar{\mathcal{P}}(X_{\tilde\sigma}), Z) = 0$ for every $Z \in T^\perp M$. Then

$$\bar{\mathcal{P}}(X_{\tilde\sigma}) \in T(J^{nh})^{-1}(\mu) \Longleftrightarrow \omega_L(\bar{\mathcal{P}}(X_{\tilde\sigma}), X_{f_i}) \circ j_M \circ j = \{\sigma, f_i\}_M \circ j = 0 \qquad 1 \leqslant i \leqslant r.$$

□

Consequently, in case we have

$$\{\sigma, f_i\}_M \circ j = 0 \qquad 1 \leqslant i \leqslant r \qquad \forall \sigma \in C^\infty_G(M) \tag{12}$$

we have proved that equality (10) holds true. Conditions (12) will not be fulfilled in general. In section 6.1.1, we will see that in the case of horizontal symmetries, $k$ is injective and conditions (12) are satisfied, and therefore, there is a well-defined (unique) almost-Poisson structure on $\overline{(J^{nh})^{-1}(\mu)}$, so that $k : \overline{(J^{nh})^{-1}(\mu)} \longrightarrow \bar M$ is an almost-Poisson morphism.

Concerning the dynamics, if $k$ is injective, then $k_*(\bar\Gamma_{L,M})_\mu = \bar\Gamma_{L,M}$. The restriction of the energy $E_L$ to $(J^{nh})^{-1}(\mu)$ is $G_\mu$-invariant, so it induces a function on $\overline{(J^{nh})^{-1}(\mu)}$, $(E_L)_\mu : \overline{(J^{nh})^{-1}(\mu)} \longrightarrow \mathbb{R}$. One can easily check that $(\bar E_L)_{|\bar M} \circ k = (E_L)_\mu$. If, in addition, (10) holds, we have that $k$ is an almost-Poisson mapping, or equivalently,

$$k_*(X^\mu_{\bar\lambda \circ k}) = X^M_{\bar\lambda} \circ k \qquad \forall\bar\lambda \in C^\infty(\bar M).$$

In particular, taking $(\bar{E}_L)_{|\bar{M}}$, we have that

$$X^{\mu}_{(E_L)_\mu}(\lambda_\mu) = X^{\mu}_{(E_L)_\mu}(\bar{\lambda} \circ k) = k_*(X^{\mu}_{(E_L)_\mu})(\bar{\lambda})$$
$$= X^{\bar{M}}_{(\bar{E}_L)_{|\bar{M}}}(\bar{\lambda}) \circ k = \bar{\Gamma}_{L,M}(\bar{\lambda}) \circ k$$
$$= k_*((\bar{\Gamma}_{L,M})_\mu)(\bar{\lambda}) = (\bar{\Gamma}_{L,M})_\mu(\lambda_\mu)$$

for all $\lambda_\mu \in C^\infty(\overline{(J^{nh})^{-1}(\mu)})$. Therefore, $X^{\mu}_{(E_L)_\mu} = (\bar{\Gamma}_{L,M})_\mu$. Then, we can conclude that the evolution of any function $\lambda_\mu \in C^\infty(\overline{(J^{nh})^{-1}(\mu)})$ along the integral curves of $(\bar{\Gamma}_{L,M})_\mu$ on $\overline{(J^{nh})^{-1}(\mu)}$ is given by

$$\dot{\lambda_\mu} = (\bar{\Gamma}_{L,M})_\mu(\lambda_\mu) = \{\lambda_\mu, (E_L)_\mu\}_\mu. \tag{13}$$

*4.1.4. The nonholonomic free particle.* Here we will discuss an instructive example due to Rosenberg [39] which has also been extensively treated in [3,4,6]. Consider a particle moving in space, so $Q = \mathbb{R}^3$, subject to the nonholonomic constraint

$$\phi = \dot{z} - y\dot{x}.$$

The Lagrangian function is

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and the Poincaré–Cartan two-form is

$$\omega_L = \mathrm{d}x \wedge \mathrm{d}\dot{x} + \mathrm{d}y \wedge \mathrm{d}\dot{y} + \mathrm{d}z \wedge \mathrm{d}\dot{z}.$$

The constraint manifold is the distribution

$$M = \left\langle \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right\rangle.$$

Consider the Lie group $G = \mathbb{R}^2$ and its action on $Q$:

$$\varphi : G \times Q \longrightarrow Q$$
$$((r, s), (x, y, z)) \longmapsto (x + r, y, z + s).$$

If we consider the lifted action $\varphi^1$ of $\varphi$ to $TQ$, given by $(\varphi^1)_g = T\varphi_g$, then the infinitesimal generators of this action are

$$\mathcal{V} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right\rangle.$$

It is simple to verify that $L$ and $M$ are $G$-invariant. Choose local coordinates $(x, y, z, \dot{x}, \dot{y})$ on $M$. We find that the distribution $F_{|M}$ is generated by the vectors fields:

$$F_{|M} = \left\langle \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \dot{x}}, \frac{\partial}{\partial \dot{y}}, \frac{\partial}{\partial \dot{z}} \right\rangle.$$

The symplectic vector bundle $F \cap TM$ is given by

$$F \cap TM = \left\langle \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \dot{x}\frac{\partial}{\partial \dot{z}}, \frac{\partial}{\partial \dot{x}} + y\frac{\partial}{\partial \dot{z}}, \frac{\partial}{\partial \dot{y}} \right\rangle$$

with symplectic orthogonal complement

$$(F \cap TM)^\perp = \left\langle \frac{\partial}{\partial \dot{z}} - y\frac{\partial}{\partial \dot{x}}, \frac{\partial}{\partial z} + \dot{x}\frac{\partial}{\partial \dot{y}} - y\frac{\partial}{\partial x} \right\rangle.$$

We realize that for each $m = (x, y, z, \dot{x}, \dot{y}) \in M$, we have

$$\mathcal{V}_m \cap F_m = \left\langle \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\rangle.$$

Therefore, we are in the general case. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and $\{e^1, e^2\}$ its dual basis. We define a section of the vector bundle $(\mathbb{R}^2)^M$:

$$\begin{aligned}\bar{\xi} : M &\longrightarrow (\mathbb{R}^2)^M \\ (x, y, z, \dot{x}, \dot{y}) &\longmapsto e_1 + ye_2.\end{aligned}$$

Its corresponding nonholonomic momentum function is

$$J^{nh}_{\bar{\xi}} = \dot{x} + y\dot{z}.$$

We can construct, from the section $\bar{\xi}$, the vector field $\Xi$:

$$\Xi = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$

Therefore, the momentum equation would be

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x} + y\dot{z}) = \dot{z}\dot{y}.$$

Using the constraint $\phi$, we may rewrite this equation as

$$\ddot{x} + \frac{y}{1+y^2}\dot{x}\dot{y} = 0. \tag{14}$$

In [3], Bates *et al* have obtained a constant of motion for this problem, apart from the energy, related with the symmetry group and the constraint. We are now going to see how the obtaining of this constant fits nicely in the geometrical setting we have exposed here.

We start by calling the nonholonomic Noether theorem [11,25,40], which ensures us when a function $\varphi$ is a constant of motion.

**Theorem 4.15.** *A function $\varphi : TQ \longrightarrow \mathbb{R}$ is a constant of the motion of $X$ if and only if the energy is constant along the integral curves of the vector field $\bar{\mathcal{P}}(X_\varphi)$, that is, $\bar{\mathcal{P}}(X_\varphi)(E_L) = 0$.*

Now, it is important to realize the following facts:

(i) $\bar{\mathcal{P}}(\Xi)(E_L) = \Xi(E_L) = 0$, because $\Xi \in F \cap TM$ and $E_L$ is $G$-invariant,
(ii) $\bar{\mathcal{P}}(X_\phi)(E_L) = \omega_L(X_{E_L}, \bar{\mathcal{P}}(X_\phi)) = -X(\phi) = 0$, because $X \in F \cap TM$.

Therefore, if we can find functions $f, g$ on $TQ$ such that the vector field $Z = f\Xi + gX_\phi$ would be Hamiltonian, say $Z = X_\varphi$, from Theorem 4.15, we would have a constant of the motion, due to the symmetry and the constraint. In general, the condition of '$Z$ to be Hamiltonian' will lead us to a quite complex first-order system of partial derivative equations. However, in this case, it is not difficult to prove (just a few computations) that $f = \frac{1}{\sqrt{1+y^2}}$ and $g = -\frac{y}{\sqrt{1+y^2}}$ are sufficient. Consequently, we obtain the conservation law

$$\varphi = \dot{x}\sqrt{1+y^2}.$$

Then we choose the following section of $(\mathbb{R}^2)^{M*} \longrightarrow Q$:

$$\begin{array}{rcccl}\mu : & Q & \longrightarrow & (\mathbb{R}^2)^{M*} & \\ & q & \longmapsto & \mu(q) : & ((\mathbb{R}^2)^q)^* \to \mathbb{R} \\ & & & & e_1 + ye_2 \mapsto c\sqrt{1+y^2}\end{array}$$

where $q = (x, y, z)$. We have that $f : M \longrightarrow \mathbb{R}$ where $f = \langle \mu, \bar{\xi}\rangle \circ \tau_Q - J^{nh}_{\bar{\xi}}$ is given by

$$f = c\sqrt{1+y^2} - \dot{x}(1+y^2).$$

The hypotheses of proposition 4.4 are fulfilled. A direct computation shows that the section $\mu$ satisfies equation (7). Then $(J^{nh})^{-1}(\mu)$ is a submanifold of $M$. In fact,

$$(J^{nh})^{-1}(\mu) = \left\{(x, y, z, \dot{x}, \dot{y})/\dot{x} = c\sqrt{1+y^2}\right\} = \{(x, y, z, \dot{y})\}.$$

As the Lie group $G = \mathbb{R}^2$ is Abelian, the coadjoint action is trivial. Then it is easily seen that the isotropy group $G_\mu$ of the action $CoAd^M$ is $G_\mu = G$. So we have the action

$$\begin{aligned} \Theta : G_\mu \times (J^{nh})^{-1}(\mu) &\longrightarrow \mathbb{R} \\ ((r, s), (x, y, z, \dot{y})) &\longmapsto (x + r, y, z + s, \dot{y}). \end{aligned}$$

Consequently, $\overline{(J^{nh})^{-1}(\mu)} = \{y, \dot{y}\}$. We obtain that

$$X_f = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} - \left(\frac{cy}{\sqrt{1+y^2}} - \dot{z}\right)\frac{\partial}{\partial \dot{y}} \in F \cap TM.$$

Therefore, for all $\sigma \in C^\infty_G(M)$, we have

$$\{\sigma, f\}_M \circ j = X^M_f(\sigma) \circ j = \mathcal{P}(X_f)(\sigma) \circ j = X_f(\sigma) \circ j = \frac{\partial \sigma}{\partial \dot{y}}\left(\frac{cy}{\sqrt{1+y^2}} - y\dot{x}\right) = 0.$$

Moreover, the mapping $k$ is injective:

$$\begin{aligned} k : \overline{(J^{nh})^{-1}(\mu)} &\longrightarrow \bar{M} \\ (y, \dot{y}) &\longmapsto (y, c\sqrt{1+y^2}, \dot{y}). \end{aligned}$$

Then, we know from the above discussion that there is a well-defined almost-Poisson structure on $\overline{(J^{nh})^{-1}(\mu)}$ which is given by

$$\{y, \dot{y}\}_\mu = 1.$$

As conditions (9) and (12) are exactly the same (due to $G_\mu = G$), we have that $\{\cdot, \cdot\}_\mu$ is the reduced bracket of $\{\cdot, \cdot\}_M$. Indeed, $\{\cdot, \cdot\}_\mu$ is integrable, that is, it is a Poisson structure.

## 5. The purely kinematic case

We now recover the discussion for general constrained systems (1) with symmetries. Suppose that $\mathcal{V}_x \cap F_x = \{0\}$ and $T_xM = \mathcal{V}_x + (F_x \cap T_xM)$, for all $x \in M$. In principle, this leads us to believe that the symmetries do not play an important role in reduction, because none of them are compatible with the bundle of reaction forces. Indeed, in this case, $\mathfrak{g}^F = 0$ and we have no constrained momentum mapping. However, we now see that the symplectic reduction explained in section 3 takes a nice form here due to the particular geometry involved in the system.

### *5.1. Reduction*

In this case, we have that $T_xM = \mathcal{V}_x \oplus (F_x \cap T_xM)$, for all $x \in M$. Moreover, $U = F \cap TM$, so $TM = \mathcal{V}_{|M} \oplus U$. Since $U$ is $G$-invariant, this decomposition defines a principal connection $\Upsilon$ on the principal $G$-bundle $\rho_{|M} : M \to \bar{M}$, with horizontal subspace $U_x$ at $x \in M$. Note, in passing, that here $U$ represents a vector bundle of constant rank. In what follows we let $X$ denote a fixed $G$-invariant solution of (1) which, moreover, belongs to $F$. In particular, this means that $X$ is horizontal, i.e. $X \in U$.

Denote by $\boldsymbol{h} : TM \longrightarrow U$ and $\boldsymbol{v} : TM \longrightarrow \mathcal{V}$ the horizontal and vertical projectors associated with the decomposition $TM = \mathcal{V}_{|M} \oplus U$, respectively. The curvature of $\Upsilon$ is the tensor field of type $(1, 2)$ on $M$ given by

$$R = \frac{1}{2}[\boldsymbol{h}, \boldsymbol{h}]$$

where [, ] denotes the Nijenhuis bracket of type $(1, 1)$ tensor fields. Taking into account that in the present case $\bar{U} = T\bar{M}$, we obtain on $\bar{M}$ a two-form $\bar{\omega}$ (which is now a genuine differential form on $\bar{M}$) and a function $\bar{H}$ such that the projection $\bar{X}$ of $X$ verifies

$$i_{\bar{X}}\bar{\omega} = \mathrm{d}\bar{H}. \tag{15}$$

It should be pointed out that the reduced two-form $\bar{\omega}$, in general, need not be closed. We will show, however, that in case the given two-form $\omega$ on $P$ is exact, one can construct a reduced equation, equivalent to (15), but now in terms of a closed two-form on $\bar{M}$.

Assume $\omega = \mathrm{d}\theta$ for some one-form $\theta$ on $P$. Denote by $\theta'$ the one-form on $M$ defined by $\theta' = j_M^*\theta$, where $j_M : M \hookrightarrow P$ is the canonical inclusion. By means of the given solution $X$ of (1) we can construct a one-form $\alpha_X$ on $M$ as follows:

$$\alpha_X = i_X(\boldsymbol{h}^*\mathrm{d}\theta' - \mathrm{d}\boldsymbol{h}^*\theta') \tag{16}$$

with the usual convention that, for an arbitrary $p$-form $\beta$, $\boldsymbol{h}^*\beta$ is the $p$-form defined by the prescription $\boldsymbol{h}^*\beta(X_1, \ldots, X_p) = \beta(\boldsymbol{h}(X_1), \ldots, \boldsymbol{h}(X_p))$.

**Proposition 5.1 ([10]).** *Assume, in addition, that the given action $\Phi$ leaves $\theta$ invariant. Then, the one-forms $\boldsymbol{h}^*\theta'$ and $\alpha_X$ are projectable. Moreover, the projection $\bar{X}$ of $X$, which is a solution of (15), also satisfies the equation*

$$i_{\bar{X}}\mathrm{d}\bar{\theta}'_h = \mathrm{d}\bar{H} - \overline{\alpha_X} \tag{17}$$

*where $\bar{\theta}'_h$ and $\overline{\alpha_X}$ are the projections of the one-forms $\boldsymbol{h}^*\theta'$ and $\alpha_X$, respectively.*

Proposition 5.1 describes a situation where a constrained Hamiltonian system (1) with symmetry, admits a reduction to an unconstrained system (17), but with an additional conservative force represented by $\overline{\alpha_X}$. Indeed, by construction, the one-form $\alpha_X$ satisfies

$$i_X\alpha_X = 0.$$

*5.1.1. Čaplygin systems.* We now consider an interesting special subcase of the purely kinematic case, namely a (generalized) Čaplygin system. For such a system, the configuration manifold $Q$ is a principal $G$-bundle $\pi : Q \longrightarrow Q/G$, and the constraints are given by the horizontal subspace of a principal connection $\gamma$ on $\pi$ (see [17, 23]). We also have a regular Lagrangian $L : TQ \longrightarrow \mathbb{R}$, which is $G$-invariant. It is known that the lifted action of $G$ on the symplectic manifold $(TQ, \omega_L)$ is Hamiltonian. Let us assume that the resulting nonholonomic system verifies the compatibility condition. The constrained equations then read as (cf (3)):

$$\begin{aligned} &i_X\omega_L - \mathrm{d}E_L \in S^*(TM^0) \\ &X_{|M} \in TM. \end{aligned} \tag{18}$$

Under the above conditions, one can easily see that there exists a well-defined Lagrangian function $L^* : T(Q/G) \longrightarrow \mathbb{R}$, given by

$$L^*(Y) = L((Y^h)_q)$$

for any $Y \in T_y(Q/G)$, where $q \in Q$ is an arbitrary point in the fibre over $y \in Q/G$ and $Y^h$ denotes the horizontal lift of $Y$ with respect to $\gamma$.

A direct computation shows that, $\mathcal{V} \cap F = \{0\}$. Moreover, we have $U = F \cap TM$, and $U$ is symplectic with respect to $\omega_L$. Therefore, we deduce that

$$TM = \mathcal{V} \oplus U.$$

Thus, a Čaplygin system indeed fits in the purely kinematic case. Moreover, one can prove that $\bar{M} = M/G \cong T(Q/G)$ and $\bar{E_L} = E_{L^*}$.

The compatibility condition $F^\perp \cap TM = 0$ ensures the existence of a unique solution $X = \Gamma_{L,M}$ of (18) which, moreover, is a SODE. Notice that $\Gamma_{L,M}$ can be obtained by projecting the unconstrained Euler–Lagrange vector field $\Gamma_L$ by means of the first projector associated with the decomposition

$$T(TQ)_{|M} = TM \oplus F^\perp.$$

Since $\omega_L = -\mathrm{d}\theta_L$, the reduced equation becomes

$$i_{\bar{X}}\omega_{L^*} = \mathrm{d}E_{L^*} - \overline{\alpha_{\Gamma_{L,M}}}$$

where $\overline{\alpha_{\Gamma_{L,M}}}$ is the projection of the one-form $\alpha_{\Gamma_{L,M}}$, defined by (16). Observe that

$$i_{\bar{\Gamma}}\overline{\alpha_{\Gamma_{L,M}}} = 0$$

for any SODE $\bar{\Gamma}$ on $T(Q/G)$. This implies that $\overline{\alpha_{\Gamma_{L,M}}}$ is a one-form of gyroscopic type.

**Remark 5.2.** As was pointed out in [38], Čaplygin considered systems with Abelian groups of symmetries and it seemed to be Voronec who extended the theory to general Lie groups.

**Remark 5.3.** After the above reduction procedure, system (17) can still possess some symmetries we have not taken into account. This is the case, for example, of *the vertical rolling disc* [6, 10]. Consider a rolling disc of radius $R$ constrained to remain vertical on a horizontal plane. The configuration space is $\mathbb{R} \times S^1 \times S^1$.

The dynamics of this mechanical system is described by

(i) the regular Lagrangian:

$$L = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2 + I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2)$$

where $m$ is the mass, and $I_1$, $I_2$ are moments of inertia;

(ii) the nonholonomic constraints:

$$\phi_1 = \dot{x} - (R\cos\theta_1)\dot{\theta}_2 = 0$$
$$\phi_2 = \dot{y} - (R\sin\theta_1)\dot{\theta}_2 = 0.$$

Consider the group $G = \mathbb{R}^2$ and its trivial action by translations on $Q$:

$$\Phi : G \times Q \longrightarrow Q$$
$$(r, s) \times (x, y, \theta_1, \theta_2) \longmapsto (x + r, y + s, \theta_1, \theta_2).$$

Note that $\rho : Q \to S^1 \times S^1$ is a principal $G$-bundle and $M$, the constraint manifold, is the horizontal sub-bundle of a principal connection, so that the given system is a Čaplygin system. Following the above analysis we then obtain

$$L^* = \frac{1}{2}(I_1\dot{\theta}_1^2 + (mR^2 + I_2)\dot{\theta}_2^2)$$
$$\omega_{L^*} = I_1\mathrm{d}\theta_1 \wedge \mathrm{d}\dot{\theta}_1 + (mR^2 + I_2)\,\mathrm{d}\theta_2 \wedge \mathrm{d}\dot{\theta}_2.$$

In this particular case the gyroscopic one-form $\overline{\alpha_{\Gamma_{L,M}}} = 0$ and $\omega_{\bar{U}} = \omega_{L^*}$. So the reduced equation (17) becomes

$$i_{\bar{X}}\omega_{L^*} = \mathrm{d}E_{L^*}.$$

Now, there are still some symmetries of this system we can consider. Denote $K$ for the Lie group $S^1 \times S^1$ and let us define

$$\begin{aligned} \Psi : K \times Q/G &\longrightarrow Q/G \\ ((\lambda_1, \lambda_2), (\theta_1, \theta_2)) &\longmapsto (\theta_1 + \lambda_1, \theta_2 + \lambda_2). \end{aligned}$$

If we consider the lifted action $\Psi^1$ of $\Psi$ to $T(Q/G)$, given by $(\Psi^1)_k = T\Psi_k$, it is clear that the Lagrangian $L^*$ is $K$-invariant. Then, we can perform further reduction.

Thus, in general, the reduced system (17) can still enjoy more symmetries to be considered. Let $\Psi : K \times \bar{M} \longrightarrow \bar{M}$ be an action on $\bar{M}$ that leaves invariant the reduced Hamiltonian $\bar{H}$ and the one-form $\bar{\theta}'_h$. We can define a momentum mapping, $J : \bar{M} \longrightarrow \mathfrak{k}^*$, in the usual manner: $\langle J(\bar{m}), \eta\rangle = -\langle \bar{\theta}'_h(\bar{m}), \eta_{\bar{M}}(\bar{m})\rangle$ for $\bar{m} \in \bar{M}$ and $\eta \in \mathfrak{k}$. It is easy to see that $i_{\eta_{\bar{M}}} \mathrm{d}\bar{\theta}'_h = \mathrm{d}J_\eta$ for all $\eta \in \mathfrak{k}$. Using equation (17) and the $K$-invariance of $\bar{H}$, we obtain a momentum equation

$$\bar{X}(J_\eta) = \overline{\alpha_X}(\eta_{\bar{M}}). \tag{19}$$

*5.1.2. The nonholonomic free particle revisited.* We will show now how the example of the nonholonomic free particle can also be seen as a Čaplygin system. With the same notations of section 4.1.4, consider the Lie group $G = \mathbb{R}$ and its trivial action by translation on $Q$:

$$\begin{aligned} \Phi : G \times Q &\longrightarrow Q \\ (s, (x, y, z)) &\longmapsto (x, y, z + s). \end{aligned}$$

Note that $M$ is the horizontal subspace of a connection $\gamma$ on the principal fibre bundle $Q \longrightarrow Q/G$, where $\gamma = (-y\,\mathrm{d}x + \mathrm{d}z)e$, with $\{e\}$ the infinitesimal generator of the translation. Therefore, this is a Čaplygin system. Following the above analysis, we obtain that

$$L^* = \tfrac{1}{2}((1 + y^2)\dot{x}^2 + \dot{y}^2)$$

and the reduced system

$$i_{\bar{X}}\omega_{L^*} = \mathrm{d}E_{L^*} - \overline{\alpha_{\Gamma_{L,M}}}$$

where $\overline{\alpha_{\Gamma_{L,M}}} = \dot{x}\dot{y}y\,\mathrm{d}x - y\dot{x}^2\,\mathrm{d}y$. Now we can take into account the remaining symmetry we have ignored so far. Consider the Lie group $K = \mathbb{R}$ and its action on $Q/G$:

$$\begin{aligned} \Psi : K \times Q/G &\longrightarrow Q/G \\ (r, (x, y)) &\longmapsto (x + r, y). \end{aligned}$$

It is clear that $L^*$ is $K$-invariant. The momentum function for this action is

$$\begin{aligned} J : T(Q/G) &\longrightarrow \mathbb{R}^* \cong \mathbb{R} \\ (x, y, \dot{x}, \dot{y}) &\longmapsto (1 + y^2)\dot{x}e. \end{aligned}$$

We compute $\overline{\alpha_{\Gamma_{L,M}}}(e_{T(Q/G)}) = \dot{x}\dot{y}y$ and using (19), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}((1 + y^2)\dot{x}) = \dot{x}\dot{y}y$$

which is just

$$\ddot{x} + \frac{y}{1 + y^2}\dot{x}\dot{y} = 0$$

that is, the same result obtained in (14).

*5.2. Reconstruction*

We now discuss the problem of reconstructing the dynamics on $M$ from the reduced dynamics on $\bar{M}$ in the case where (1) admits a unique solution $X$. Suppose the flow of the reduced system $\bar{X}$ is known. Take $\bar{c}(t)$ an integral curve of $\bar{X}$ starting from a point $\bar{x} \in \bar{M}$, and fix $x \in \rho^{-1}(\bar{x})$. We want to find the corresponding integral curve $c(t)$ of $X$ starting from $x$ which projects on $\bar{c}(t)$, i.e. $\rho(c(t)) = \bar{c}(t)$. But we must realize that the curve $c(t)$ is just the horizontal lift of $\bar{c}(t)$ starting from $x$ with respect to the principal connection $\Upsilon$. We prove this simple fact in the following.

**Proposition 5.4.** *The curve $c(t)$ is the horizontal lift of $\bar{c}(t)$ starting from $x$ with respect to the principal connection $\Upsilon$.*

**Proof.** Let $d(t)$ denote the horizontal lift of $\bar{c}(t)$ starting from $x$. Therefore, $\rho(d(t)) = \bar{c}(t)$ and $d(0) = x$. Since $X$ and $\bar{X}$ are $\rho$-related, we have that $T\rho(X(d(t))) = \bar{X}\rho(d(t)) = \bar{X}\bar{c}(t) = \dot{\bar{c}}(t) = T\rho(\dot{d}(t))$. Therefore, $\dot{d}(t) - X(d(t))$ is vertical. But it is also horizontal, because $X \in U$. Then we deduce that $\dot{d}(t) = X(d(t))$. □

Thus, in the vertical case, the reconstruction problem is just a horizontal lift operation. We now briefly recall the concepts of *geometric, dynamic and total phases* for the reconstruction process [34]. The geometric phase is just the holonomy of the path $\bar{c}(t)$ with respect to the connection $\Upsilon$, that is, the Lie group element $g$ so that $d(1) = g \cdot d(0)$. In general, we will have that $c(t)$, the integral curve projecting on $\bar{c}(t)$, is not exactly $d(t)$, the horizontal lift of $\bar{c}(t)$, but a shift of this curve, $c(t) = g(t) \cdot d(t)$. We call the Lie group element $g(1)$ the dynamic phase, and the total phase will stand for $h = g(1) \cdot g$.

**Corollary 5.5.** *In the vertical case, the geometric phase coincides with the total phase.*

*5.2.1. Čaplygin systems.* Concerning the reconstruction process for Čaplygin systems, the above description remains valid, of course, but we can say a little more about the holonomy of the two connections, $\gamma$ and $\Upsilon$. The following diagram will be helpful:

$$\begin{array}{ccc} TM = U \oplus \mathcal{V}_\rho & \longrightarrow & T\bar{M} \cong \bar{U} \\ \downarrow & & \downarrow \\ TQ = M \oplus \mathcal{V}_\pi & \longrightarrow & T(Q/G) \cong \bar{M} \\ \downarrow & & \downarrow \\ Q & \longrightarrow & Q/G. \end{array}$$

Let $\bar{c}(t)$ be the integral curve of $\bar{X}$ starting from $\bar{x}$. Fix $x \in \rho^{-1}(\bar{x})$ and consider its horizontal lift, $c(t)$, with respect to $\Upsilon$ starting from $x$. We have proved that $c(t)$ is precisely the integral curve of $X$ starting from $x$ which projects on $\bar{c}(t)$. Let $\bar{q}(t)$ be the projection of $\bar{c}(t)$ to $Q/G$, $\bar{q}(t) = \pi_{Q/G}(\bar{c}(t))$. We denote by $q^M(t)$ its horizontal lift with respect to $\gamma$. Finally, we write $q(t) = \pi_Q(c(t))$. Then we have $\pi(q(t)) = \pi \circ \pi_Q(c(t)) = \pi_{Q/G} \circ \rho(c(t)) = \pi_{Q/G}(\bar{c}(t)) = \bar{q}(t)$. Since $c(t)$ is an integral curve of a SODE, we have $c(t) = \dot{q}(t) \in M$. So we have proved that $q(t)$ is just the horizontal lift of $\bar{q}(t)$, i.e. $q(t) = q^M(t)$.

Now, we study the holonomy of $\bar{c}(t)$. Let us suppose that $\bar{c}(t)$ is a closed loop. We have $\bar{c}(0) = \bar{c}(1) = \bar{x}$ and $c(0) = x$. Consequently, $c(1) = gx$ and $g$ is the geometric phase, which is, in the vertical case, the total phase. As $c(t) = \dot{q}^M(t)$, we have that $\dot{q}^M(1) = g\dot{q}^M(0)$ which in particular implies that $q^M(1) = gq^M(0)$. We have then proved the result of the following proposition.

**Proposition 5.6.** *The geometric phase (respect to $\Upsilon$) of a closed integral curve of $\bar{X}$ is the same as the geometric phase (respect to $\gamma$) of its projection to $Q/G$.*

*5.2.2. Plate with a knife edge on an inclined plane.* The configuration space of the plate with a knife edge on an inclined plane is $Q = \mathbb{R}^2 \times S^1$ with coordinates $(x, y, \theta)$ (see, for example, [38] for more details). This system is determined by the following data:

- the regular Lagrangian function $L$:

$$\begin{aligned} L : TQ &\longrightarrow \mathbb{R} \\ (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) &\longmapsto \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}k^2\dot{\theta}^2 + gx\sin\alpha \end{aligned}$$

  where the mass of the plate is assumed equal to unity;
- the nonholonomic constraint function:

$$\phi = \dot{y} - \dot{x}\tan\theta.$$

Consider the Lie group $G = \mathbb{R}$ and its trivial action by translation on $Q$:

$$\begin{aligned} \Phi : \mathbb{R} \times Q &\longrightarrow Q \\ (r, (x, y, \theta)) &\longmapsto (x, y + r, \theta) \end{aligned}$$

with associated fibration

$$\begin{aligned} \rho : Q &\longrightarrow \mathbb{R} \times S^1 \\ (x, y, \theta) &\longmapsto (x, \theta). \end{aligned}$$

Note that $\rho : Q \longrightarrow \mathbb{R} \times S^1$ is a principal bundle, with structure group $G$, and $M$, the constraint submanifold, is the horizontal distribution of a principal connection, $\gamma$. The connection one-form is $\gamma = \mathrm{d}y - \tan\theta\,\mathrm{d}x$. Therefore, this is a Čaplygin system.

The corresponding reduced system (17) is described by:

- the reduced Lagrangian $L^*$:

$$\begin{aligned} L^* : T(\mathbb{R} \times S^1) &\longrightarrow \mathbb{R} \\ (x, \theta, \dot{x}, \dot{\theta}) &\longmapsto \tfrac{1}{2}(\sec^2\theta\,\dot{x}^2 + k^2\dot{\theta}^2) + gx\sin\alpha \end{aligned}$$

- the gyroscopic one-form:

$$\overline{\alpha_{\Gamma_{L,M}}} = \tan\theta\sec^2\theta[(\dot{x})^2\,\mathrm{d}\theta - \dot{x}\dot{\theta}\,\mathrm{d}x].$$

After some calculations, one finds the following equations of motion:

$$\begin{aligned} \ddot{x} &= -\dot{x}\dot{\theta}\tan\theta + g\sin\alpha\cos^2\theta \\ \ddot{\theta} &= 0. \end{aligned}$$

We obtain that $\theta = \omega t + \theta_0$, where $\omega$ and $\theta_0$ are constants. Consequently, a solution for the initial conditions $\theta_0 = x_0 = \dot{x}_0 = 0$ and $\dot{\theta}_0 = \omega$ is

$$\begin{aligned} x &= \frac{g}{2\omega^2}\sin\alpha\sin^2\omega t \\ \theta &= \omega t. \end{aligned}$$

This curve $\bar{q}(t) = (x(t), \theta(t))$ is closed since

$$\bar{q}(0) = \bar{q}(2\pi/\omega).$$

The horizontal lift $q(t) = q^M(t)$ of the curve $\bar{q}(t)$ with initial conditions $\theta_0 = x_0 = \dot{x}_0 = y_0 = \dot{y}_0 = 0$ and $\dot{\theta}_0 = \omega$ is

$$\begin{aligned} x &= \frac{g}{2\omega^2}\sin\alpha\sin^2\omega t \\ y &= \frac{g}{2\omega^2}\sin\alpha[\omega t - \tfrac{1}{2}\sin\omega t] \\ \theta &= \omega t. \end{aligned}$$

Observe that $q(0) = (0, 0, 0)$ and $q(2\pi/\omega) = (0, \frac{g\pi}{\omega^2}\sin\alpha, 0)$. Therefore, the geometric phase of the curve $\bar{q}(t)$ is $\frac{g\pi}{\omega^2}\sin\alpha$.

## 6. The case of horizontal symmetries

The assumption now is that $\mathcal{V}_x \cap F_x = \mathcal{V}_x$, for all $x \in M$ or, equivalently, $\mathcal{V}_{|M} \subset F$. In particular, every infinitesimal generator of the given group action then yields a horizontal symmetry. Thus, in this case, all the symmetries are compatible with the bundle $F$. This leads us to suspect that we can perform a holonomic-type reduction. Note, also, that an unconstrained Hamiltonian system with symmetry can be regarded as a special subcase of this case, since we then have $M = P$, $F = TP$ and, obviously, $\mathcal{V} \subset TP$.

### *6.1. Reduction*

For the further analysis of this case we assume, in addition, that the given symplectic action $\Phi$ on $P$ is a Hamiltonian action, in the sense that it admits an $Ad^*$-equivariant momentum mapping $J : P \longrightarrow \mathfrak{g}^*$, such that for all $\xi \in \mathfrak{g}$, $i_{\xi_P}\omega = \mathrm{d}\langle J, \xi\rangle$. It follows from the definition of the momentum mapping that $\xi_P = X_{J_\xi}$, where $J_\xi(x) = J(x)(\xi)$ for all $x \in P$. Taking into account that, by assumption, $\mathcal{V}_{|M} \subset F$, we find that for any solution $X$ of (1), along the constraint submanifold $M$ we have

$$X(J_\xi) = 0$$

i.e., the components of the momentum mapping are conserved quantities for the constrained dynamics. This is a version of Noether's theorem for constrained systems. (For the case of mechanical systems with nonholonomic constraints, see, in this respect, [6, 11, 40].)

Let $\mu \in \mathfrak{g}^*$ be a regular value of $J$. Since the action, $\Phi$, of $G$ on $P$ is free and proper, we have that the isotropy group $G_\mu$ acts freely and properly on the level set $J^{-1}(\mu)$. It is known (see [1,30,36,37]) that under these conditions $(P_\mu = J^{-1}(\mu)/G_\mu, \omega_\mu)$ is a symplectic manifold, where $\omega_\mu$ is the two-form defined by

$$\pi_\mu^*\omega_\mu = j_\mu^*\omega$$

with $\pi_\mu : J^{-1}(\mu) \longrightarrow P_\mu$ the canonical projection, and $j_\mu : J^{-1}(\mu) \hookrightarrow P$ the natural inclusion.

Imposing a condition of clean intersection of $M$ and $J^{-1}(\mu)$, we have that $M' = M \cap J^{-1}(\mu)$ is a submanifold of $J^{-1}(\mu)$ which is $G_\mu$-invariant. Passing to the quotient we then obtain a submanifold $M_\mu = M'/G_\mu$ of $P_\mu$ (that, with the adequate embedding, can be identified with $\bar{M} \cap P_\mu$). Next, we can define a distribution $F'$ on $P$ along $M'$ by putting

$$F'_{x'} = T_{x'}(J^{-1}(\mu)) \cap F_{x'} \qquad \forall x' \in M'$$

and we now make the further simplifying assumption that $F'$ has constant rank. It is obvious that $F'$ is a $G_\mu$-invariant sub-bundle of $TP_{|M'}$ and, hence, it projects onto a sub-bundle $F_\mu$ of $TP_\mu$ along $M_\mu$. Finally, since the restriction of the Hamiltonian $H$ to $J^{-1}(\mu)$ is also $G_\mu$-invariant, it induces a function $H_\mu$ on $P_\mu$.

**Theorem 6.1 ([10]).** *Suppose that $X$ is a $G$-invariant solution of (1). Then, $X$ induces a vector field $X_\mu$ on $M_\mu$, such that*

$$\begin{aligned} &(i_{X_\mu}\omega_\mu - \mathrm{d}H_\mu)_{|M_\mu} \in F_\mu^0 \\ &X_\mu \in TM_\mu. \end{aligned} \tag{20}$$

In the case of horizontal symmetries we have thus proved that, under the appropriate assumptions, the given constrained problem on $(P, \omega)$ reduces to a constrained problem on $(P_\mu, \omega_\mu)$.

*6.1.1. Lagrangian systems.* Let us suppose that we have a nonholonomic Lagrangian system which fulfils the compatibility condition. Next, we show that $k : \overline{(J^{nh})^{-1}(\mu)} \longrightarrow \bar{M}$ is injective and conditions (12), obtained in section 4.1.3 to perform a kind of reduction via the mapping $k$, are satisfied.

**Proposition 6.2.** *Let $k : \overline{(J^{nh})^{-1}(\mu)} = M_\mu \longrightarrow \bar{M}$ be the composition of $k_\mu : M_\mu \longrightarrow M/G_\mu$ and $p : M/G_\mu \longrightarrow \bar{M}$. Then we can define on $M_\mu$ a unique almost-Poisson structure so that $k$ is an almost-Poisson mapping.*

**Proof.** It is an easy exercise to prove that $k$ is injective in the case of horizontal symmetries. From the analysis of section 4.1.3, we know that it is sufficient to prove conditions (12). Now, taking $\xi_1, \ldots, \xi_r$ as a base of the Lie algebra $\mathfrak{g}$, we have that $f_i = \langle \mu, \xi_i \rangle - J_{\xi_i}$, $1 \leqslant i \leqslant r$. Given $\sigma \in C^\infty_G(M)$, we deduce that

$$\{\sigma, f_i\}_M \circ j = (\xi_i)_{TQ}(\tilde{\sigma}) \circ j_M \circ j = (\xi_i)_M(\sigma) \circ j = 0$$

due to the $G$-invariance of $\sigma$. □

On the other hand, we have that the symplectic distribution $F \cap TM$ induces a symplectic distribution $F_\mu \cap TM_\mu$ in $TP_\mu = T(TQ)_\mu$, that is

$$T(TQ)_{\mu|M_\mu} = (F_\mu \cap TM_\mu) \oplus (F_\mu \cap TM_\mu)^{\perp_\mu}$$

with induced projectors for each $\bar{v}_q \in M_\mu$

$$\bar{\mathcal{P}}_\mu : T_{\bar{v}_q}(TQ)_\mu \longrightarrow ((F_\mu)_{\bar{v}_q} \cap T_{\bar{v}_q} M_\mu) \qquad \bar{\mathcal{Q}}_\mu : T_{\bar{v}_q}(TQ)_\mu \longrightarrow ((F_\mu)_{\bar{v}_q} \cap T_{\bar{v}_q} M_\mu)^{\perp_\mu}.$$

The above descomposition induces an almost-Poisson bracket $\{\cdot, \cdot\}_{M_\mu}$ on $M_\mu$, in the same manner as we previously did for $M$ in section 4.1.2. More precisely, given $\lambda_\mu, \sigma_\mu : M_\mu \longrightarrow \mathbb{R}$, take $\tilde{\lambda}_\mu$, $\tilde{\sigma}_\mu$ arbitrary extensions to $(TQ)_\mu$, $\tilde{\lambda}_\mu \circ j_{M_\mu} = \lambda_\mu$, $\tilde{\sigma}_\mu \circ j_{M_\mu} = \sigma_\mu$, with $j_{M_\mu} : M_\mu \hookrightarrow (TQ)_\mu$, and define

$$\{\lambda_\mu, \sigma_\mu\}_{M_\mu} = (\omega_L)_\mu(\bar{\mathcal{P}}_\mu(X^\mu_{\tilde{\lambda}_\mu}), \bar{\mathcal{P}}_\mu(X^\mu_{\tilde{\sigma}_\mu})) \circ j_{M_\mu}.$$

Indeed, we have that $\{\cdot, \cdot\}_{M_\mu} = \{\cdot, \cdot\}_\mu$, as we prove in the following.

**Theorem 6.3.** *Consider $(M_\mu, \{\cdot, \cdot\}_{M_\mu})$ and $(\bar{M}, \{\cdot, \cdot\}_{\bar{M}})$. Then $k : M_\mu \longrightarrow \bar{M}$ is an almost-Poisson mapping.*

**Proof.** First of all, consider the following commutative diagrams:

$$\begin{array}{ccc} (J^{nh})^{-1}(\mu) = M' & \xrightarrow{j} & M \\ i \downarrow & & \downarrow j_M \\ J^{-1}(\mu) & \xrightarrow{j_\mu} & TQ \end{array} \qquad \begin{array}{ccc} M' & \xrightarrow{i} & J^{-1}(\mu) \\ \pi_{M'} \downarrow & & \downarrow \pi_\mu \\ M_\mu & \xrightarrow{j_{M_\mu}} & (TQ)_\mu. \end{array}$$

Now, the proof is a careful exercise of equalities. Indeed, given $\lambda_\mu, \sigma_\mu : M_\mu \longrightarrow \mathbb{R}$, we have

$$\begin{aligned} \{\lambda_\mu, \sigma_\mu\}_\mu \circ \pi_{M'} &= \{\bar{\lambda}, \bar{\sigma}\}_{\bar{M}} \circ k \circ \pi_{M'} = \{\lambda, \sigma\}_M \circ j = \omega_L(\bar{\mathcal{P}}(X_{\tilde{\lambda}}), \bar{\mathcal{P}}(X_{\tilde{\sigma}})) \circ j_M \circ j \\ &= (j_M \circ j)^* \omega_L(\bar{\mathcal{P}}(X_{\tilde{\lambda}}), \bar{\mathcal{P}}(X_{\tilde{\sigma}})) = (\pi_\mu \circ i)^*(\omega_L)_\mu(\bar{\mathcal{P}}(X_{\tilde{\lambda}}), \bar{\mathcal{P}}(X_{\tilde{\sigma}})) \\ &= (\omega_L)_\mu(\bar{\mathcal{P}}_\mu(X^\mu_{\tilde{\lambda}_\mu}), \bar{\mathcal{P}}_\mu(X^\mu_{\tilde{\sigma}_\mu})) \circ j_{M_\mu} \circ \pi_{M'} = \{\lambda_\mu, \sigma_\mu\}_{M\mu} \circ \pi_{M'}. \end{aligned}$$

□

**Remark 6.4.** It should be noted that from the general discussion in section 4.1, it is concluded that for nonholonomic Lagrangian systems which fit in the horizontal case, theorem 6.3 is the utmost one can say. That is, while conditions (12) are always fulfilled, conditions (9) are no longer satisfied in general. This means, in particular, that the almost-Poisson bracket $\{\cdot,\cdot\}_{M_\mu}$ is not the reduced bracket of $\{\cdot,\cdot\}_M$, as it was stated in [9] (theorem 8.2). However, following (13), we know that for all $\lambda_\mu \in C^\infty(M_\mu)$, its evolution along the integral curves of the dynamics is given by

$$\dot{\lambda_\mu} = (\Gamma_{L,M})_\mu(\lambda_\mu) = \{\lambda_\mu, (E_L)_\mu\}_{M_\mu}.$$

*6.2. Reconstruction*

As far as the reconstruction of the dynamics is concerned, we observe that, unlike in the purely kinematic case, we first have to select an arbitrary connection on the principal $G_\mu$-bundle $M' \longrightarrow M_\mu$. This connection will enable us to subsequently lift the integral curves of the reduced system from $M_\mu$ to $M'$.

More precisely, let $\Upsilon$ be such a principal connection. We start with $c_\mu(t)$, an integral curve of $X_\mu$ with the initial condition $c_\mu(0) = m_\mu$, $m_\mu \in M_\mu$. We choose $m \in (\pi_\mu)^{-1}(m_\mu)$ and wish to find the unique integral curve $c(t)$ of $X$ which satisfies $c(0) = m$. As $X$ and $X_\mu$ are $\pi_\mu$-related, $c(t)$ projects on $c_\mu(t)$. We will proceed in a similar way as in Marsden *et al* [34] for the holonomic reconstruction.

Consider $d(t)$ the horizontal lift of $c_\mu(t)$ with $d(0) = m$, that is, $\pi_\mu(d(t)) = c_\mu(t)$ and $\Upsilon(\dot{d}(t)) = 0$. Put $c(t) = g(t)d(t)$, for some curve $g(t)$ in $G_\mu$, with $g(0) = e$. As $c(t)$ is an integral curve of $X$, we have that $X(c(t)) = \dot{c}(t)$, i.e.,

$$X(g(t)d(t)) = (g(t)\dot{d(t)}) = g(t)\dot{d}(t) + g(t)((g^{-1}(t)\dot{g}(t))_M d(t)).$$

As $X(g(t)d(t)) = g(t)X(d(t))$, we conclude

$$X(d(t)) = \dot{d}(t) + (g^{-1}(t)\dot{g}(t))_M d(t). \tag{21}$$

So we can factorize the reconstruction process in two steps:

(i) To find a curve $\xi(t)$ in $\mathfrak{g}_\mu$ so that

$$\xi(t)_M(d(t)) = X(d(t)) - \dot{d}(t).$$

(ii) To find a curve $g(t)$ in $G_\mu$ so that

$$\dot{g}(t) = g(t)\xi(t) \qquad g(0) = e.$$

Making use of the connection $\Upsilon$, we can replace (i) by

$$(\mathrm{i}')\ \xi(t) = \Upsilon(\xi(t)_M(d(t))) = \Upsilon(X(d(t)) - \dot{d}(t)) = \Upsilon(X(d(t))).$$

*Cotangent bundles.* We now discuss the case in which $P = T^*Q$, and $G$ acts freely on $Q$, $\phi : G \times Q \longrightarrow Q$, and therefore on $P$ by cotangent lift, $\Phi : G \times P \longrightarrow P$. We will show below that if the bundle $\varsigma_\mu : Q \longrightarrow Q/G_\mu$ has a connection, for a certain $\mu$ to be specified, this induces a connection on $\rho : M' \longrightarrow M_\mu$.

The momentum mapping $J : T^*Q \longrightarrow \mathfrak{g}^*$ for the Hamiltonian action $\Phi$ is defined by $\langle J(\alpha_q), \xi\rangle = \langle \alpha_q, \xi_Q(q)\rangle = \langle \theta(\alpha_q), \xi_{T^*Q}(\alpha_q)\rangle$, where $\alpha_q \in T_q^*Q$, $\xi \in \mathfrak{g}$. Let $\mu \in \mathfrak{g}^*$ be a regular value of $J$ and suppose again that its isotropy group $G_\mu$ acts freely and properly on the level set $J^{-1}(\mu)$. As before, we consider the symplectic manifold $(T^*Q)_\mu = (J^{-1}(\mu)/G_\mu, \omega_\mu)$. We denote by $\mu' = \mu_{|\mathfrak{g}_\mu} \in \mathfrak{g}_\mu^*$, the restriction of $\mu$ to $\mathfrak{g}_\mu$. Assume $\varsigma_\mu : Q \longrightarrow Q/G_\mu$ is a principal $G$-bundle and let $\gamma \in \Lambda^1(Q, \mathfrak{g}_\mu)$ be a connection form on it. We recall now the cotangent bundle reduction theorem of Satzer, Marsden and Kummer (see [1, 21]).

**Theorem 6.5.** *Let $\Omega$ be the curvature of $\gamma$ and let $B$ the pull-back by $\tau : T^*(Q/G_\mu) \longrightarrow Q/G_\mu$ of the closed two-form on $Q/G_\mu$ induced by the $\mu'$-component of $\Omega$, $\mu' \cdot \Omega \in \Lambda^2(Q)$. Endow $T^*(Q/G_\mu)$ with the symplectic form $\omega - B$, where $\omega$ is the canonical two-form of the cotangent bundle. Then $(T^*Q)_\mu$ is symplectically embedded in $(T^*(Q/G_\mu), \omega - B)$ and its image is a vector sub-bundle with base $Q/G_\mu$. This embedding is onto if and only if $\mathfrak{g} = \mathfrak{g}_\mu$.*

The following conmutative diagram will help us to handle the theorem:

$$\begin{array}{ccccccccccc}
M' & \longrightarrow & J^{-1}(\mu) & \longrightarrow & J_\mu^{-1}(\mu') & \xrightarrow{t_\mu} & J_\mu^{-1}(0) & \longrightarrow & T^*Q & \xrightarrow{\tau_Q} & Q \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
M_\mu & \longrightarrow & (T^*Q)_\mu & & \xrightarrow{\varphi_\mu} & & T^*(Q/G_\mu) & & \xrightarrow{\tau_{Q/G_\mu}} & & Q/G_\mu
\end{array}$$

where $J_\mu : T^*Q \longrightarrow \mathfrak{g}_\mu^*$, $t_\mu : J_\mu^{-1}(\mu') \longrightarrow J_\mu^{-1}(0)$ and $\varphi_\mu : (T^*Q)_\mu \longrightarrow T^*(Q/G_\mu)$ are respectively defined by $J_\mu(\alpha_q) = J(\alpha_q)_{|\mathfrak{g}_\mu}$, $t_\mu(\alpha_q) = \alpha_q - \mu' \cdot \gamma_q(\cdot)$ and $\varphi_\mu(\bar{\alpha_q}) = \overline{\alpha_q - \mu' \cdot \gamma_q(\cdot)}$ for all $\alpha_q \in T^*Q$.

The connection $\gamma \in \Lambda^1(Q, \mathfrak{g}_\mu)$ induces a connection $\Upsilon \in \Lambda^1(M', \mathfrak{g}_\mu)$ by pull-back, $\Upsilon = (\tau_Q \cdot t_\mu)^*\gamma$ so that $\Upsilon_{\alpha_q}(U_{\alpha_q}) = \gamma_q(T\tau_Q \cdot U_{\alpha_q})$ for all $U_{\alpha_q} \in T_{\alpha_q}M'$. Thus now, we can rewrite (i′) above as

$$\text{(i}'\text{)}\ \xi(t) = \Upsilon(X(d(t))) = \gamma(T\tau_Q \cdot X(d(t))) = \gamma(FH(d(t)))$$

where $FH : T^*Q \longrightarrow TQ$ is the fibre derivative of the Hamiltonian $H : T^*Q \longrightarrow \mathbb{R}$.

*6.2.1. Lagrangian systems.* If we have a Lagrangian of mechanical type, $L = T - V$, where $T$ is the kinetic energy of a Riemannian metric $g$ on $Q$, and $V$ is a potential energy, we know (see [23]) that the nonholonomic Lagrangian system fulfils the compatibility condition. Making use of the metric $g$, we can define a natural connection, to be called the *mechanical connection*, on the principal fibre bundle $\varsigma_\mu : Q \longrightarrow Q/G_\mu$ as follows: we take $\mathcal{V}_{\varsigma_\mu} = \ker T\varsigma_\mu$ and consider $\mathcal{H} = \mathcal{V}_{\varsigma_\mu}^\perp$, the orthogonal complement of $\mathcal{V}_{\varsigma_\mu}$ with respect to the metric $g$. We define $\gamma_{\rm mech}$ as the connection on $Q \longrightarrow Q/G_\mu$ whose horizontal subspace is $\mathcal{H}$.

We know that $FH(\alpha_q) = \alpha_q^\sharp$, where $H$ is defined from $E_L$ through the Legendre transformation and $\sharp$ denotes the natural pairing of vectors and co-vectors of $Q$ induced by the metric $g$. Again, we can rewrite $(i')$ in the following form:

$$\text{(i}'\text{)}\ \xi(t) = \gamma_{\rm mech}(q(t))(FH(d(t))) = \gamma_{\rm mech}(q(t))(d(t)^\sharp)$$

with $q(t) = \tau(d(t))$.

If we define for each $q \in Q$ the *$\mu$-locked inertia tensor* (see [33]), $I_\mu(q) : \mathfrak{g}_\mu \longrightarrow \mathfrak{g}_\mu^*$, by $\langle I_\mu(q)\zeta, \eta\rangle = \langle \zeta_Q(q), \eta_Q(q)\rangle$, we can verify $\gamma_{\rm mech}(v_q) = I_\mu^{-1}(q)J(v_q^\flat)$, with $v_q^b$ the co-vector associated to $v_q$ through the metric. We then rewrite (i′) as,

$$\text{(i}'\text{)}\ \xi(t) = \gamma_{\rm mech}(q(t))(d(t)^\sharp) = I_\mu^{-1}(q(t))(\mu).$$

Compare this result with those in [6].

## 7. A special subcase of the general case

Now we are going to consider the case in which the bundle $\mathfrak{g}^F$ is trivial, that is, $\mathfrak{g}^x = \mathfrak{g}_0$, $\forall x \in M$. Following [9], we can prove the following proposition.

**Proposition 7.1.** *$\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$ which is invariant with respect to the adjoint representation.*

Next we consider $G_0$, the normal connected subgroup of $G$ with Lie algebra $\mathfrak{g}_0$ and $\Phi_0 : G_0 \times P \longrightarrow P$, the restricted action to $G_0$. For this action, it is clear that $\mathcal{V}_{0|M} \subset F \cap TM$, so we are in the case of horizontal symmetries. Now we are going to proceed in the way described above.

As before, we can assume that $\Phi_0$ on $P$ is a Hamiltonian action, this is, it admits an $Ad^*$-equivariant momentum mapping $J : P \longrightarrow \mathfrak{g}_0^*$, such that for all $\xi \in \mathfrak{g}_0$, $i_{\xi_P}\omega = \mathrm{d}J_\xi$. Let $\mu \in \mathfrak{g}_0^*$ be a regular value of $J$ and suppose that $G_\mu^0$, its isotropy group in $G_0$, acts freely and properly on the level set $J^{-1}(\mu)$. Under these conditions, $(P_\mu = J^{-1}(\mu)/G_\mu^0, \omega_\mu)$ is a symplectic manifold. We also suppose that $M$ and $J^{-1}(\mu)$ have a clean intersection, $M' = M \cap J^{-1}(\mu)$, which is a $G_\mu^0$-invariant submanifold of $J^{-1}(\mu)$. We then consider $M_\mu = M'/G_\mu^0$. We can define a distribution $F'$ on $P$ along $M'$ by putting $F'_{x'} = T_{x'}(J^{-1}(\mu)) \cap F_{x'}, \forall x' \in M'$ and in addition assume that $F'$ has constant rank. Again, $F'$ is $G_\mu^0$-invariant and it projects onto a sub-bundle $F_\mu$ of $TP_\mu$ along $M_\mu$. Finally, with the function $H_\mu$ induced by the restriction of the Hamiltonian $H$ to $J^{-1}(\mu)$, we have all the ingredients to apply theorem 6.1 and obtain the following reduced constrained problem on $(P_\mu, \omega_\mu)$:

$$\begin{array}{l} (i_{X_\mu}\omega_\mu - \mathrm{d}H_\mu)_{|M_\mu} \in F_\mu^0 \\ X_\mu \in TM_\mu. \end{array} \qquad (22)$$

So far, we have reduced the constrained problem by the horizontal symmetries and have again obtained a constrained problem. We will now investigate what happens with the symmetries we have not used yet. In the following, we are going to take them into account.

For this purpose, we consider the action $\Psi : G_\mu \cdot G_0/G_0 \times P_\mu \longrightarrow P_\mu$ defined by $\Psi(\bar{g}, \bar{p}) = \overline{\Phi(g, p)}$. Note that this action is well defined because we are not treating with all the remaining symmetries $G/G_0$, but only with the adequate ones to $P_\mu$. Indeed, we prove the following lemma.

**Lemma 7.2.** *The mapping $\Psi$ is well defined.*

**Proof.** We must verify that given $\bar{g}, \bar{h} \in G_\mu \cdot G_0/G_0$ and $\bar{p}, \bar{q} \in P_\mu$ so that $\bar{g} = \bar{h}$ and $\bar{p} = \bar{q}$, we have $\Psi(\bar{g}, \bar{p}) = \Psi(\bar{h}, \bar{q})$. Since $G_\mu \cdot G_0/G_0 \cong G_\mu/G_\mu \cap G_0 = G_\mu/G_\mu^0$, we can consider $\bar{g}, \bar{h}$ as elements of this latter group, so we have that $h^{-1}g \in G_\mu \cap G_0$. We also have that there exists $i \in G_\mu^0$ such that $p = iq$. Then $gp = giq = gih^{-1}hq$. Moreover, $gih^{-1} = (ih^{-1}g)^{g^{-1}} \in G_0$, because $i$ and $h^{-1}g$ are in $G_0$, and this group is normal in $G$. Clearly $gih^{-1} \in G_\mu$, so finally we have that $gih^{-1} \in G_\mu^0$. We have obtained $\overline{gp} = \overline{hq}$, i.e., $\Psi(\bar{g}, \bar{p}) = \Psi(\bar{h}, \bar{q})$. □

In a similar way, we can check easily that $\Psi$ is a symplectic action on $P_\mu$ and that $M_\mu$, $F_\mu$ and $H_\mu$ are all $G_\mu/G_\mu^0$-invariant. We denote $\rho_\mu : P_\mu \longrightarrow \bar{P}_\mu$ the canonical projection for $\Psi$ and $\mathcal{V}_\mu = \ker T\rho_\mu$.

Our aim is to prove that, under the assumption $TM_\mu = (F_\mu \cap TM_\mu) + \mathcal{V}_{\mu|M_\mu}$, the constrained Hamiltonian problem with symmetries on $(P_\mu, \omega_\mu)$ fits in the purely kinematic case. For this purpose, we now identify the fundamental vector fields for the action $\Psi$.

**Lemma 7.3.** *Let $\zeta + \mathfrak{g}_\mu \cap \mathfrak{g}_0$ be an element of $\mathfrak{g}_\mu/\mathfrak{g}_\mu \cap \mathfrak{g}_0$, the Lie algebra of $G_\mu/G_\mu^0$. Then*

$$(\zeta + \mathfrak{g}_\mu \cap \mathfrak{g}_0)_{P_\mu}(\bar{p}) = T\pi_\mu \zeta_{J^{-1}(\mu)}(p) \qquad \forall p \in J^{-1}(\mu)$$

*where $\pi_\mu : J^{-1}(\mu) \longrightarrow P_\mu$ is the projection mapping associated to the action of $G_\mu^0$ on $J^{-1}(\mu)$ and $\zeta_{J^{-1}(\mu)}$ is the fundamental vector field corresponding to the action of $G_\mu$ on $J^{-1}(\mu)$.*

**Proof.** We have

$$(\zeta + \mathfrak{g}_\mu \cap \mathfrak{g}_0)_{P_\mu}(\bar{p}) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|t=0} \Psi(\exp(t\zeta + \mathfrak{g}_\mu \cap \mathfrak{g}_0), \bar{p}) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|t=0} \Psi(\overline{\exp_\mu t\zeta}, \bar{p})$$
$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{|t=0} (\overline{\exp_\mu t\zeta \cdot p}) = T\pi_\mu \zeta_{J^{-1}(\mu)}(p).$$

□

Now, we are in a position to prove the former statement.

**Proposition 7.4.** *If $TM_\mu = (F_\mu \cap TM_\mu) + \mathcal{V}_{\mu|M_\mu}$, the reduced constrained Hamiltonian system (22), considered with the action $\Psi$ on $P_\mu$, fits in the purely kinematic case.*

**Proof.** We must prove that $(\mathcal{V}_\mu)_{\bar{x}} \cap (F_\mu)_{\bar{x}} = \{0\}, \forall \bar{x} \in M_\mu$. Suppose that $(\zeta + \mathfrak{g}_\mu \cap \mathfrak{g}_0)_{P_\mu}(\bar{x}) \in (F_\mu)_{\bar{x}}$ for some $\bar{x} \in M_\mu$. Recall that $F_{\mu\bar{x}} = T\pi_\mu F'_x$. Then, we have that there exists $Y \in F'_x$ such that $T\pi_\mu(Y) = (\zeta + \mathfrak{g}_\mu \cap \mathfrak{g}_0)_{P_\mu}(\bar{x}) = T\pi_\mu(\zeta_{J^{-1}(\mu)}(x))$ which, in turn, implies there exists $\xi \in \mathfrak{g}^0_\mu = \mathfrak{g}_\mu \cap \mathfrak{g}_0$ such that $\zeta_{J^{-1}(\mu)}(x) = Y + \xi_{J^{-1}(\mu)}(x)$. Therefore, $(\zeta - \xi)_{J^{-1}(\mu)}(x) = Y$, which gives $\zeta - \xi \in \mathfrak{g}^x = \mathfrak{g}_0$. Obviously, $\zeta - \xi \in \mathfrak{g}_\mu$. Then, $\zeta + \mathfrak{g}_\mu \cap \mathfrak{g}_0 = \xi + \mathfrak{g}_\mu \cap \mathfrak{g}_0 = 0 + \mathfrak{g}_\mu \cap \mathfrak{g}_0$. □

Next, we proceed as in section 5.1. We obtain a principal connection $\Upsilon$ on the principal $(G_\mu/G^0_\mu)$-bundle $\rho_{\mu|M_\mu} : M_\mu \longrightarrow \bar{M}_\mu$, with horizontal subspace $U_{\bar{x}} = (F_\mu)_{\bar{x}} \cap T_{\bar{x}} M_\mu$ at each point $\bar{x} \in M_\mu$.

If we assume again that $(P, \omega)$ is an exact symplectic manifold, with $\omega = \mathrm{d}\theta$, we can define in a natural manner $\theta_\mu$ so that $\omega_\mu = \mathrm{d}\theta_\mu$. Obviously, $\theta_\mu$ is $G_\mu/G^0_\mu$-invariant. Let $\theta'_\mu = j^*_{M_\mu}\theta_\mu$, where $j_{M_\mu} : M_\mu \hookrightarrow P_\mu$ is the canonical inclusion. Then proposition 5.1 applies to the reduced constrained Hamiltonian problem (22) to give

$$i_{\bar{X}_\mu}\bar{\omega} = \mathrm{d}\bar{H}_\mu - \overline{\alpha_{X_\mu}} \tag{23}$$

where $\overline{\alpha_{X_\mu}}$ is the projection of $\alpha_{X_\mu}$, with $\alpha_{X_\mu} = i_{X_\mu}(\boldsymbol{h}^*\,\mathrm{d}\theta'_\mu - \mathrm{d}\boldsymbol{h}^*\theta'_\mu)$, and $\bar{\omega} = \mathrm{d}(\bar{\theta}'_\mu)_h$, with $(\bar{\theta}'_\mu)_h$ the projection of $\boldsymbol{h}^*\theta_\mu$.

**Remark 7.5.** In general, the condition '$\mathfrak{g}^x$ does not depend on $x \in M$' seems to be quite restrictive. In [40], Śniatycki defined $\mathfrak{g}' \subset \mathfrak{g}$ by

$$\mathfrak{g}' = \{\xi' \in \mathfrak{g} \mid \text{there exists a constant section } \bar{\xi} \text{ of } \mathfrak{g}^F \text{ with } \bar{\xi}(x) = \xi', \forall x \in M\}.$$

In other words, $\mathfrak{g}'$ consists of those elements of $\mathfrak{g}$ such that its corresponding infinitesimal generator of the induced action on $M$ is a horizontal symmetry. If $\mathfrak{g}^x$ does not depend on $x \in M$, it is clear that $\mathfrak{g}_0 = \mathfrak{g}'$.

Śniatycki claims that $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$ and then he considers the normal connected subgroup $G'$ of $G$ with Lie algebra $\mathfrak{g}'$. The reduction process is parallel to the one performed here until we reach proposition 7.4, which will not be true in general.

As in the case of reduction, reconstruction of the dynamics is a two-step process: first, implementation of a purely kinematic-type reconstruction and then of a horizontal-type one.

### *7.1. The nonholonomic free particle modified*

Next we are going to treat the example of the nonholonomic free particle, but with a different constraint. As before, we have a particle moving in space, subject to the nonholonomic constraint

$$\phi = \dot{z} - x\dot{x}.$$

This constraint is semi-holonomic [2] and, consequently, the problem admits an unconstrained description on the leaves of the foliation defined on $Q$. Anyway, we will ignore this point, just to illustrate the two-step reduction procedure developed above.

The Lagrangian function is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and the constraint submanifold is defined through the distribution

$$M = \left\langle \frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right\rangle.$$

Consider the Lie group $G = \mathbb{R}^2$ and its action on $Q$:

$$\begin{aligned} \varphi : G \times Q &\longrightarrow Q \\ ((r,s),(x,y,z)) &\longmapsto (x, y+r, z+s). \end{aligned}$$

If we consider the lifted action $\Phi$ of $\varphi$ to $TQ$, given by $\Phi_g = T\varphi_g$, then the infinitesimal generators of this action are

$$\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}.$$

It is a simple verification to see that $L$ and $M$ are $G$-invariant. Choose local coordinates $(x, y, z, \dot{x}, \dot{y})$ on $M$. We find that the distribution $F_{|M}$ is generated by the vectors fields:

$$\left\{ \frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \dot{x}}, \frac{\partial}{\partial \dot{y}}, \frac{\partial}{\partial \dot{z}} \right\}.$$

We realize that for each $m = (x, y, z, \dot{x}, \dot{y}) \in M$, we have

$$\mathcal{V}_m \cap F_m = \left\langle \frac{\partial}{\partial y} \right\rangle.$$

Note that the fibre $(\mathbb{R}^2)^m$ does not depend on the base point $m \in M$. Then, the bundle $(\mathbb{R}^2)^F$ is trivial and we are just in the special subcase of the general case treated in this section. With the notations we have been using, $\mathfrak{g}_0 = \mathbb{R} \times \{0\}$ and $G_0 = \mathbb{R} \times \{0\}$. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and $\{e^1, e^2\}$ its dual basis. Now, consider $\Phi_0$, the restricted action of $\Phi$ to $G_0$. $\Phi_0$ is Hamiltonian, with momentum mapping:

$$\begin{aligned} J : T\mathbb{R}^3 &\longrightarrow \mathbb{R}^* \\ (x, y, z, \dot{x}, \dot{y}, \dot{z}) &\longmapsto \dot{y}e^1. \end{aligned}$$

Let $\mu = ae^1 \in \mathbb{R}^*$. We have that $G^0_\mu = \mathbb{R}$ and $J^{-1}(\mu) = \{(x, y, x, \dot{x}, \dot{z})\}$. Therefore,

$$(T\mathbb{R}^3)_\mu = \{(x, z, \dot{x}, \dot{z})\} \qquad (\omega_L)_\mu = \mathrm{d}x \wedge \mathrm{d}\dot{x} + \mathrm{d}z \wedge \mathrm{d}\dot{z}.$$

We note that $M$ and $J^{-1}(\mu)$ have a clean intersection $M' = \{(x, y, z, \dot{x})\}$ so that

$$M_\mu = \{(x, z, \dot{x})\}.$$

After some computations, we find that

$$F_\mu = \left\langle \frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \frac{\partial}{\partial \dot{x}}, \frac{\partial}{\partial \dot{z}} \right\rangle$$

$$F_\mu \cap TM_\mu = \left\langle \frac{\partial}{\partial \dot{x}} + x\frac{\partial}{\partial \dot{z}}, \frac{\partial}{\partial x} + x\frac{\partial}{\partial z} + \dot{x}\frac{\partial}{\partial \dot{z}} \right\rangle.$$

Finally, we obtain $(E_L)_\mu = \frac{1}{2}(\dot{x}^2 + \dot{z}^2 + a^2)$. With all these ingredients, we pose the following constrained problem (22) on $((T\mathbb{R}^3, (\omega_L)_\mu))$:

$$\begin{aligned} &(i_{(\Gamma_{L,M})_\mu}(\omega_L)_\mu - \mathrm{d}(E_L)_\mu)_{|M_\mu} \in F^0_\mu \\ &(\Gamma_{L,M})_\mu \in TM_\mu. \end{aligned} \tag{24}$$

Now, we investigate what happens with the symmetries we have not used yet. We have that $G_\mu = \mathbb{R}^2$ and consequently, $(G_\mu + G_0)/G_0 \cong \mathbb{R}$. Then we consider the action

$$\begin{aligned} \Psi : (G_\mu + G_0)/G_0 \times (T\mathbb{R}^3)_\mu &\longrightarrow (T\mathbb{R}^3)_\mu \\ (s, (x, z, \dot{x}, \dot{z})) &\longmapsto (x, z + s, \dot{x}, \dot{z}). \end{aligned}$$

The canonical projection $\rho_\mu$ is given by

$$\begin{aligned} \rho_\mu : (T\mathbb{R}^3)_\mu &\longrightarrow \overline{(T\mathbb{R}^3)}_\mu \\ (x, z, \dot{x}, \dot{z}) &\longmapsto (x, \dot{x}, \dot{z}) \end{aligned}$$

and its restriction to $M_\mu$ is

$$\begin{aligned} \rho_{\mu_{|M_\mu}} : M_\mu &\longrightarrow \bar{M}_\mu \\ (x, z, \dot{x}) &\longmapsto (x, \dot{x}). \end{aligned}$$

The vertical bundle of the action $\Psi$ is

$$\mathcal{V}_\mu = \left\langle \frac{\partial}{\partial z} \right\rangle.$$

For each $m_\mu \in M_\mu$ we have that

$$(\mathcal{V}_\mu)_{m_\mu} \cap (F_\mu)_{m_\mu} = \{0\}.$$

Moreover, $TM_\mu = F_{\mu_{|M_\mu}} \cap TM_\mu + \mathcal{V}_{\mu_{|M_\mu}}$. Therefore, the constrained system (24) on $((T\mathbb{R}^3)_\mu, (\omega_L)_\mu)$ fits in the purely kinematic case, that is, we obtain a principal connection $\Upsilon$ on the principal $\mathbb{R}$-bundle $\rho_{\mu_{|M_\mu}} : M_\mu \longrightarrow \bar{M}_\mu$, with horizontal subspace $U_{m_\mu} = (F_\mu)_{m_\mu} \cap T_{m_\mu} M_\mu$ at each point $m_\mu \in M_\mu$. The connection one-form is

$$\Upsilon = (\mathrm{d}z)e$$

where $\{e\}$ is the canonical basis of the Lie algebra $(\mathfrak{g}_\mu + \mathfrak{g}_0)/\mathfrak{g}_0 \cong \mathbb{R}$. We have that $(T\mathbb{R}^3, \omega_L)$ is an exact symplectic manifold, so we can define

$$\theta_\mu = -\dot{x}\,\mathrm{d}x - \dot{z}\,\mathrm{d}z$$

and $(\omega_L)_\mu = \mathrm{d}\theta_\mu$. We check that $\theta'_\mu = j^*_{M_\mu}\theta_\mu = -\dot{x}(\mathrm{d}x + x\,\mathrm{d}z)$. Next, we calculate the one-form $\alpha_{(\Gamma_{L,M})_\mu}$ on $M_\mu$ defined by the prescription $\alpha_{(\Gamma_{L,M})_\mu} = i_{(\Gamma_{L,M})_\mu}(\boldsymbol{h}^*\mathrm{d}\theta'_\mu - \mathrm{d}\boldsymbol{h}^*\theta'_\mu)$. First, we have that

$$\boldsymbol{h}^*\mathrm{d}\theta'_\mu = \mathrm{d}\boldsymbol{h}^*\theta'_\mu = (1 + x^2)\,\mathrm{d}\dot{x} \wedge \mathrm{d}x$$

and consequently, $\alpha_{(\Gamma_{L,M})_\mu} = 0$. Projecting onto $\bar{M}_\mu$, we obtain that

$$\begin{aligned} \bar{\omega} &= (1 + x^2)\,\mathrm{d}x \wedge \mathrm{d}\dot{x} \\ \overline{(E_L)}_\mu &= \tfrac{1}{2}(\dot{x}^2(1 + x^2) + a^2). \end{aligned}$$

Now, following (23), we can write, from the constrained problem (24), the reduced unconstrained system

$$i_{\overline{(\Gamma_{L,M})}_\mu}\bar{\omega} = \mathrm{d}\overline{(E_L)}_\mu. \tag{25}$$

From a straightforward computation we have that the solution $\overline{(\Gamma_{L,M})}_\mu$ of equation (25) is the vector field

$$\overline{(\Gamma_{L,M})}_\mu = \dot{x}\frac{\partial}{\partial x} - \frac{x\dot{x}^2}{1 + x^2}\frac{\partial}{\partial \dot{x}}.$$

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